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CORRECTION

Attention is called to an error in the abstract of "A Theory of the Stimulus" by J. A. Lynch, pages 313-314, Psychometrika, December, 1940:

The expression reading, "the integral of", should be, "x is a function of" what follows.

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THE DIFFICULTY OF A TEST AND ITS FACTOR COMPOSITION

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A factor analysis of the ten sub-tests of the Seashore test of pitch discrimination revealed that more than one ability is involved. One factor, which accounted for the greater share of the variances, had loadings that decreased systematically with increasing difficulty. A second factor had strongest loadings among the more difficult items, particularly those with frequency differences of 2 to 5 cycles per second. A third had strongest loadings at differences of 5 to 12 cycles per second. No explanation for the three factors is apparent, but the hypothesis is accepted that they represent distinct abilities. In tests so homogeneous as to content and form, where a single common factor might well have been expected, the appearance of additional common factors emphasizes the importance of considering the difficulty level of test items, both in the attempt to interpret new factors and in the practice of testing. The same kind of item may measure different abilities according as it is easy or difficult for the individuals to whom it is applied.

Several considerations have been responsible for leading the writer into this brief exploratory study of the relation of the factor composition of a test to its level of difficulty. In a previous study of difficulty (2), it was found that the ease with which an item is passed does not increase fully as rapidly as the physical magnitude of the stimulus or stimulus difference, but increases in proportion to the logarithm of that magnitude. One hypothesis suggested to account for this result assumed that different abilities are brought into play when items are easy as compared with the situation when items are difficult. The kinds of items in question were those in the Seashore tests of musical talent, namely, discrimination of pitch, intensity, and time, and memory for musical melodies. Responses to these items for the most part involve judgments similar to those of the classical psychophysical type, and judgments of this kind are often suspected of resting upon different kinds of criteria. Some criteria may be of greater relative importance when items are easy as compared with when they are difficult. One objective approach to the discovery of these criteria should be that of factor analysis. There are probably a number of ways in which factor analysis could be applied in this connection. One approach is to regard the judgments from the point of view of their

correctness as in scoring a test and to correlate the test scores so obtained.

Elsewhere the writer has pointed out the importance of considering the relative difficulty of test items for the tested population when interpreting factors revealed by statistical analysis (3). The work of Hertzman (4) had already demonstrated possible changes in factor pattern as difficulty of the same kind of item is altered. Studies of Woodrow (5) on changes in factor pattern as a result of practice also point to the same result, for one consequence of practice is to alter the proportion of correct responses and therefore to change the difficulty of the items, as we conceive of difficulty empirically defined and measured.

Another consideration prompting the factor analysis of the pitch-discrimination tests arose in connection with attempts to improve the scoring of the Seashore tests. It was hoped that the reliability and validity of these tests could be increased by introducing weighted scoring of the parts. One approach to this end was to assume a single *g* factor, to intercorrelate the parts, determine the correlation of each part with the *g* factor, and then to set up regression weights on this basis. But it was found that Spearman's criterion of a *g* factor was not satisfied, particularly in the case of the pitch test. In order to examine the matter more fully, since more than one common factor was evident, it was necessary to make a multiple-factor analysis. The centroid method was accordingly applied.

The Analysis

The data consisted of the responses of 300 University of Nebraska students to the 100 items in the Seashore test of pitch discrimination as a regular laboratory exercise. Each of the ten parts, of ten items each, was scored separately, the scores having a possible range of 0 to 10 points. Because of the chance factor this range was restricted, usually to one of 5 to 10 points. Because the easiest parts yielded so many perfect scores, the range was often further restricted and many of the distributions were decidedly skewed. It was assumed that the distribution of true scores should be normal on every test, and so to eliminate the effects of restricted range of scores and skewed distributions tetrachoric correlations were computed. For every test the distribution of scores was divided into two categories, splitting the population as close to the median as circumstances would permit. The most lopsided division was a .10 and .90 apportionment, placing all scores of 9 points and below in the lower group and all the scores of 10 in the higher group. The coefficients were estimated by means

of the Thurstone diagrams (1)*. The intercorrelations are given in Table 1.

TABLE 1

Correlation Matrix of the Ten Sub-Tests in Seashore's Test of Pitch Discrimination

Differences in cycles per second	30	23	17	12	8	5	3	2	1	Scaled Difficulty
30										—1.38
23	.79									—1.18
17	.76	.73								—0.96
12	.66	.69	.78							—0.71
8	.62	.72	.78	.84						—0.42
5	.46	.36	.50	.50	.62					—0.07
3	.30	.49	.35	.38	.50	.60				0.30
2	.37	.38	.55	.29	.42	.50	.52			0.59
1	—.19	—.12	.07	.01	.11	.30	.30	.12		1.10
0.5	—.22	—.17	—.13	—.07	—.12	—.08	.03	—.02	.13	1.60

Table 1 is worthy of comment in itself. The systematic arrangement of the coefficients in this matrix is apparent and is prognostic of relative simplicity of factorial structure. One general tendency is for items of similar difficulty to correlate more highly with one another than with dissimilar items. Another is for decreasing correlations, in general, with increasing difficulty of item throughout the table. The most difficult part, in fact, even correlates negatively with almost all other parts, and the next most difficult part has two negative correlations, these being with the two easiest parts. On the face of it, if the easier items are true measures of an ability known as pitch discrimination, then the most difficult items certainly are not. The most strongly correlated parts are those with frequency differences of 8 and 12 cycles per second ($r = .84$).

The extraction of factors by the centroid method was carried through four stages. Criteria of when to stop extracting factors led to somewhat conflicting conclusions, some indicating that two factors were sufficient and others that as many as four were significant. Rotation of the four-factor system gave a reduction to three significant factors with a fourth weak doublet, so it was decided that three were all that should be given credence. Table 2 presents the centroid loadings for the first three centroid factors.

After making graphic rotations in each of the planes, three new reference axes were located yielding the new factor loadings as shown in Table 3. The goal in rotation was to maximize the number of zero

*I am indebted to N. Y. A. assistance for the statistical work through this stage.

TABLE 2

Centroid Factor Loadings in the Ten Sub-Tests of the Seashore Test of Pitch Discrimination

Stimulus Difference	k_1	k_2	k_3	Communality
30	.76	.41	—.19	.781
23	.79	.35	—.29	.837
17	.86	.24	—.05	.800
12	.80	.29	.21	.774
8	.88	.13	.26	.864
5	.72	—.36	.19	.685
3	.63	—.45	—.15	.622
2	.59	—.29	—.32	.528
1	.12	—.47	.18	.271
0.5	—.14	—.19	—.00	.058

TABLE 3

Factor Loadings after Rotation to Maximize the Number of Zero Loadings

Stimulus Difference	I	II	III	Communality
30	.88	—.01	—.01	.780
23	.91	.10	—.08	.838
17	.86	.13	.20	.803
12	.78	—.04	.40	.772
8	.77	.11	.51	.859
5	.42	.50	.51	.684
3	.37	.67	.20	.625
2	.44	.58	—.01	.528
1	—.15	.38	.32	.273
0.5	—.21	.12	—.04	.058

loadings and at the same time to reach a positive manifold. With the exception of two minor negative loadings in the two most difficult parts, both the goal criteria were readily realized with orthogonal rotations. Loadings greater than .20 are probably significant.

The systematic arrangement of the factor loadings is most apparent when displayed graphically as in Fig. 1. The loadings in the diagrams differ slightly from those in Table 3 because they were estimated graphically directly from the axes of apparent best fit, whereas those in the table were computed after measuring the angles of rotation to the nearest degree. There are only trivial discrepancies between the two.

It can be seen that Factor I, with only minor deviations, has load-

ings that are inversely proportional to the difficulty of the item. Difficulty of items is plotted here in terms of a rational scaling based upon the total proportion of failures (corrected for chance success) for each stimulus difference. The scale values are inversely proportional to the logarithms of the stimulus differences (2). Factor II

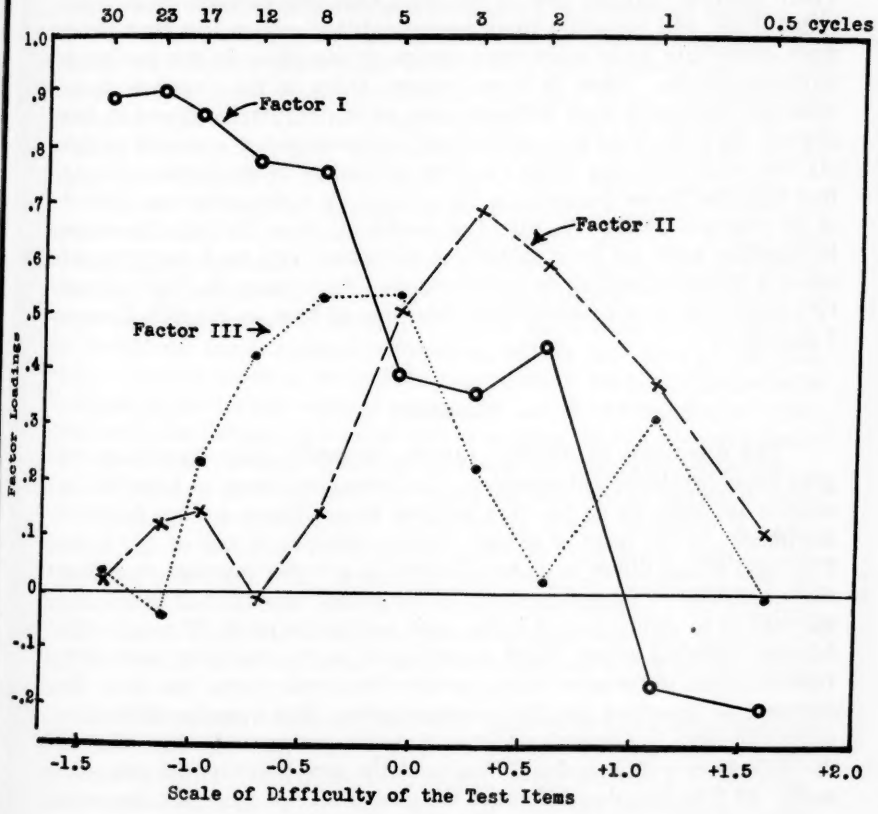

FIGURE 1

has significant loadings in items 1, 2, 3, and 5 (given in terms of frequency differences), with a maximum at 3. It is a striking fact that the distribution of loadings (if we may call it a distribution) is unimodal and quite regular in contour. Factor III is also regular and uni-

modal in its relation to difficulty, if we ignore the exceptional loadings at stimulus difference I. Its maximum is in the region of 5 to 8, and it has significant loadings probably all the way from 3 to 17 cycles per second.

At this point it may be well to raise the question of the suitability of leaving the two most difficult items in the analytical picture. Their intercorrelations are all so small that one suspects that their reliabilities, and certainly their communalities, are so low that these tests contribute little more than errors of sampling to the resulting analytic picture. What is more serious, there is the question as to whether they could have induced some of the regularity found in the system. In order that this point might be investigated, a second analysis was made omitting these two test variables. It was reassuring to find that the factor pattern for the remaining eight tests was altered in no essential detail. We may feel confident, then, that the two tests in question have not been statistical disturbers and have not brought about a fictitious regularity in the results. As appears in Fig. 1, those two tests merely fall consistently into line, at least as regards factors I and II.

Discussion

The discovery of distinct factors naturally prompts efforts to give them psychological meaning. Unfortunately there is little information available to us for this purpose here. There are no familiar landmarks in the form of already known abilities in any of the tests. The tests do not differ, as is usually true in a factor problem, in either their material or formal content. They are all composed of tones and the task is to compare and judge each pair as to pitch. The only difference, existing among them superficially is the changing pitch difference, some differences being greater than and others less than the differential threshold for the average testee. The average difference threshold here is approximately at 5 cycles per second, since that is the difference with a scaled difficulty at the zero point on the rational scale. At this stimulus difference the probability of a correct response (corrected for chance) is .50. In general, the relationship between a test and the population examined by it, as the writer has pointed out elsewhere (3), may have much to do with the manner in which individuals go to work upon its items. Their manner of working on a test is its functional aspect. The functional aspect of a test, unlike its material and formal aspects, depends upon the individual as well as upon the test and the relation between the two.

The kind of test under consideration has long been known as a

pitch-discrimination test. Pitch is a unitary variable of tones, and it has been customarily assumed that each individual has one ability to make discriminations among tones on the basis of this important variable. In making these assumptions the great complexity of the mechanisms lying between the sound waves and the expressed judgment has been overlooked. It is not to be denied that a unitary function, which when measured for goodness of performance becomes an ability, may possibly rest upon a very complicated mechanism. But if any mechanism underlying what is to direct observation a single function has parts whose efficiencies can vary at all independently, then there is room for the single function to be less efficient in one respect than in another. Statistical treatment of measured output may then break down the function into smaller functional unities.

The act of comparing tones as to pitch and rendering judgments can be logically broken down by armchair job analysis. It includes, first of all, the preparatory set of the observer, including his motivation and his adjustment of his auditory mechanisms for favorable reception of the particular stimuli to come. More than one ability may be important for individual differences within this part of the task alone. Second, there is the auditory component, proper. This includes the hearing of the two tones in succession and the production of whatever auditory pattern two tones like this make for the observer under the set of the moment. Third, it involves the language component—the connection between auditory patterns and motor patterns from which issues the expression of a judgment. If one thinks of the anatomical parts involved, the entire task seems even more complicated. Even the inner ear with its basilar membrane, so important for the auditory phase, is now known to respond differentially to different frequencies. Different parts or different separations may have varying efficiences, as is shown by the experiments on artificial tone deafness.

It has probably been the belief of those who have employed the Seashore test that individual differences in scores have meant rather purely and simply something with regard to the efficiency of the auditory mechanisms alone. It has been felt that the individuals tested are approximately on a par with respect to the other items mentioned as parts of the entire task as outlined above. The faith in these assumptions may be justified; on the other hand, individual differences in those non-auditory activities may be reflected in the test-score differences. And they may be reflected to different degrees for items of varying difficulty. For example, differences in attentional set, or of some aspect of it, for the particular kind of listening required in this kind of test may make no difference whatever in a person's score when

items are easy, but those differences in attention may be of considerable importance when the judging is more exacting.

If the three factors apparent in Fig. 1 do refer to the three main parts of the entire task of judging pitch differences, it would be hard to say which of the three is actually the auditory component, which is attentional, and which verbal. It may be that the entire task does not break up along these three lines when statistical analysis is applied to the data. And if it does not, which kind of analysis, the statistical or the armchair variety, gives the more significant components for scientific descriptive purposes? The "armchair" components are based upon many years of psychological research, yet they may be lacking in full descriptive usefulness. The statistically-derived components, if new, may point to descriptive categories of improved usefulness. If the three factors, on the other hand are confined to the auditory component, we must ask whether there are three functional mechanisms, perhaps within the inner ear, which can vary independently in efficiency in such a manner that the judgments of large, medium, and small differences depend upon them to varying degrees.

These are only a few of the questions raised by the factorial results; there are many others. Some have their origin from purely statistical considerations. The behavior of Factor I in particular might cause one to suspect that some arbitrary characteristic of the data may have induced the factor pattern. Could the increasing chance factor in judging the stimuli as difficulty increases have been responsible for the decreasing factor loadings in Factor I? The increasing chance factor causes decreasing reliability of scores, and also decreasing communality. The reliabilities of scores for the ten short tests were not estimated; they would of course exceed the communalities, which are known. The communalities do not decrease continuously from the easiest to the most difficult items, but have a maximum at some intermediate level of difficulty. The factor loadings in Factor I do not, therefore, fall off in perfect correlation with communality, although there is a strong correlation between them. As a rough test of the matter, the writer computed for every test the percentage of the variance contributed to the total communality by each factor. The result was that when these percentages are plotted against difficulty the factor picture is much the same as in Fig. 1. From this rather incomplete examination of the matter, it would seem that varying reliability, or at least varying communality, is not to blame for the pattern with respect to Factor I. It is difficult to see how varying reliability could produce anything like Factors II and III.

Other statistical incidents that may have had their influence upon the factor pattern consist in the manner of scoring and of correlat-

ing scores. The use of tetrachoric correlations was previously defended. In fact, it would have been well nigh impossible to defend the correlation of raw scores by Pearson's product-moment method. If tetrachoric r's are independent of the point of division of data, within the limits of lopsidedness which had to be tolerated in this study, there is no basis for thinking that the varying points of division had anything to do with the introduction of the factors revealed by analysis.

Lacking any extraneous way of accounting for the very systematic factors emerging from the data, we may accept the hypothesis that they represent three distinct human abilities involved in the comparison and judgment of tones as to pitch. Remembering that the judgments are like most psychophysical judgments and therefore dependent upon a variety of criteria, it may be found that these abilities represent the degrees of efficiency of using different criteria or sets of criteria. Further experimental work coupled with factorial methods is needed in order to identify such abilities or criteria of judgment. Such investigations should bring into use populations of higher and lower average scores in these tests, also populations of greater and less homogeneity. They should introduce stimuli from different parts of the entire pitch scale. New factors might be introduced, but the expected systematic variation of these and the old factors with the conditions of the experiments may suggest their real nature. Tests of differing nature, with tones and other content, will probably be required to lend more certainty to the identity of factors.

If the ability hypothesis is correct, the implications for further factor-analysis investigations of all sorts and for the practices of mental testing are of greatest importance. The interpretation of factors found in all tests should certainly take into consideration the relative difficulty of a test for the tested population. The finding of three distinct factors in tests differing neither in material nor in formal content but only in difficulty makes one wonder whether or not some of the factors already found, though seeming to tally with clusters of tests differing in material or formal content, really have their right to distinction only on the basis of difficulty. It is more likely, however, that we need merely to recognize the universal rule that changing the difficulty of a test for a population alters its functional content—the manner in which testees work upon it.

It is to be expected that by the experimental use of tests varying in difficulty general laws of shifting importance of factors will be unearthed. For example, the same kind of test item that demands inductive and/or deductive processes when difficult may require only good perceptive efficiency when easy. Differences in attentional set may be of insignificant importance when items are difficult but may

play a role when they are easy. It is possible, for example, that the Factor I of this study is merely a matter of one kind of attentional set. Easy discriminations, to follow this hypothesis, are missed largely because of lapses of attention, whereas more difficult items are missed for other reasons, and attentional lapses become comparatively insignificant.

In testing practice the implication is clear that one must adapt the test to the population or it may not measure what we think it measures. For the population examined in this study, for example, if Factor I is the one we wish to measure we should certainly not include the two most difficult parts. If Factor II is the ability we wish to measure we should most certainly eliminate the four easiest parts and perhaps the most difficult. For a population of greater or less average talent (but of comparable range) a still different selection of tests should be made. Which one of the factors is most pertinent for musical talent or even for that limited area known as pitch discrimination? This is a matter for empirical determination. Perhaps all factors are important, and perhaps, after all, a composite score derived from all levels of difficulty, except the hardest in this case, will be most predictive of the area of talent known as musical.

Quite commonly, in test practice, it is found that items of median difficulty are the most valid. They can make the most discriminations among individuals. From Fig. 1, however, we note that such items may be the least pure. Items with a stimulus difference of 5 cycles have significant and almost equal loadings in all three factors, though they do not have the highest communality. It may be that with practical criteria the more impure or composite tests factorially are usually more valid than are pure tests, for practical criteria are usually composite. This does not detract from the importance of the factor approach. The factorial composition of criteria themselves should often be determined as an efficient means of searching for tests that will predict them. But the substance of this paragraph is that test items of medium difficulty are not necessarily factorially pure or simple. From what we have seen here they may be just the opposite.

A final word will be appreciated by all those with an operationist's bent. The naming of any test in terms of its supposed functional aspect would seem to be a most questionable procedure. Since the functions tested depend upon the relation of the test to the testees and consequently what they actually do with the items, the same test cannot accurately be described by the same functional name at all times. If only one of the three factors found in this investigation deserved to be called "pitch discrimination," for example, only certain levels of difficulty of item would come under that appellation, and then

only for the kind of population under consideration. The name "pitch difference" test as a general designation might have been much better, leaving more exact psychological names to be applied later when and if necessary. It is needless to point out that the names of tests have been a constant source of confusion and the cause of much needless labor in the past. We can and should free ourselves from similar mistakes in the future.

REFERENCES

1. Chesire, L., Saffir, M., and Thurstone, L. L. Computing diagrams for the tetrachoric correlation coefficient. Chicago: Univ. Chicago Book Store, 1933.
2. Guilford, J. P. The psychophysics of mental test difficulty. *Psychometrika*, 1937, **2**, 121-133.
3. Guilford, J. P. Human abilities. *Psychol. Rev.* 1940, **47**, 367-394.
4. Hertzman, M. The effects of the relative difficulty of mental tests on patterns of mental organization. *Arch. Psychol.*, 1936, No. 197.
5. Woodrow, H. The effect of practice on test intercorrelations. *J. educ. Psychol.*, 1938, **29**, 561-572.

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A NON-GRAPHICAL METHOD FOR TRANSFORMING AN ARBITRARY FACTOR MATRIX INTO A SIMPLE STRUCTURE FACTOR MATRIX

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The most commonly used method of factoring a matrix of intercorrelations is the centroid method developed by L. L. Thurstone. It is, however, necessary to transform the centroid matrix of factor loadings into a simple structure matrix in order to facilitate the interpretation of the factor loadings. Current methods for effecting this transformation are chiefly graphical and require considerable experience and personal judgment. This paper presents a new method for transforming an arbitrary factor matrix into a simple structure matrix by methods almost completely objective. The theory underlying the method is developed and approximation procedures are derived. The method is applied to a matrix of factor loadings previously analyzed by Thurstone.

Various methods are available for determining the factor loadings of a battery of tests, from the matrix of intercorrelations or covariances among them. There has been no general agreement as to which of the methods is best. It is generally agreed however that the factor matrix should be such as to enable one most accurately to reproduce the correlation matrix with the minimum number of factors. This is no more than the application of the principle of parsimony applied to the description of a test battery. But to any given factor matrix an infinite number of non-singular transformations may be applied which in no way affect the accuracy with which the original correlation matrix may be reproduced.*

Thurstone has developed the concept of "simple structure" as a criterion for determining the best transformation for an arbitrary factor matrix. According to this concept each column of factor loadings in the factor matrix should have a minimum number of negative loadings and a maximum number of loadings which are approximately zero. Similarly each row should have a maximum number of approximately zero loadings with only a small number of positive loadings. This concept is primarily the application of the principle of parsimony

* We shall not here consider how the correlation matrix is reproduced by the factor matrix, as this relationship is extensively discussed in numerous published works.

to the description of each individual test in the battery. Each test is to be described in terms of as few factors as possible.

Thurstone (3) and his students have developed ingenious graphical and mechanical procedures for transforming "centroid," "principal axis," and arbitrary factor matrices into factor matrices satisfying within specified limits the simple structure criteria. These methods require considerable experience, skill, and judgment in their application, and other investigators have not been notably successful in their attempts to satisfy the "simple structure" criteria.

Tucker (4) has reported progress on a method which begins by assuming zero loadings in a given factor for certain of the tests. By analytical methods the best values for the "zero" and "non-zero" loadings are then determined. By means of graphical procedures certain tests are shifted from one to the other of the two categories and the values recalculated by analytical methods. This process is continued until a satisfactory simple structure is obtained. The method has not yet been developed to the point where it is considered sufficiently satisfactory to replace the graphical and mechanical procedures.

In this paper we shall describe a method which bears some resemblance to Tucker's method. However, the method involves no graphical operations and is almost completely objective.

1. Analytical Formulation

We begin with Thurstone's principle that each column of the factor matrix should have a minimum number of negative values and a maximum number of nearly vanishing values. We state this condition analytically as follows: For a given factor the sum of the squares of the significant* factor loadings divided by the sum of the squares of all the loadings shall be a maximum. We let

a — an arbitrary factor loading matrix whose m columns correspond to factors and whose n rows correspond to tests.

c — an $m \times m$ matrix which transforms the arbitrary matrix a into the simple structure matrix.

b — the simple structure matrix.

c_i — the i'th column of c corresponding to the ith column of b.

a_{l_i} — a matrix made up from all the rows of a whose corresponding tests have significant factor loadings in factor i of the simple structure matrix b.

* The term "significant" should here be interpreted in its common non-technical rather than its statistical sense, although a rigorous operational definition is implicit in the subsequent discussion.

The condition which we wish to satisfy can then be expressed analytically as follows.

$$\frac{c_i' a_{l_i}' a_{l_i} c_i}{c_i' a' a c_i} = K = \text{a maximum.} \tag{1}$$

We shall assume for the moment that we have a satisfactory method for selecting the rows in a_{l_i} from a. Our problem then is to determine c_i so that K in (1) is a maximum.

Taking logarithms of both sides of (1), we have

$$\log K = \log (c_i' a_{l_i}' a_{l_i} c_i) - \log (c_i' a' a c_i) . \tag{2}$$

Taking differentials of both sides of (2) gives

$$\frac{dK}{K} = \frac{d(c_i' a_{l_i}' a_{l_i} c_i)}{c_i' a_{l_i}' a_{l_i} c_i} - \frac{d(c_i' a' a c_i)}{c_i' a' a c_i} . \tag{3}$$

From (3)

$$\frac{1}{2K}\frac{dK}{dc'} = \frac{a_{l_i}' a_{l_i} c_i}{c_i' a_{l_i}' a_{l_i} c_i} - \frac{a' a c_i}{c_i' a' a c'} . \tag{4}$$

After equating (4) to zero, we get

$$a_{l_i}' a_{l_i} c_i = K a' a c_i . \tag{5}$$

Premultiplying both sides of (5) by $(a' a)^{-1}$ gives

$$(a' a)^{-1} a_{l_i}' a_{l_i} c_i = K c_i \tag{6}$$

or

$$[(a' a)^{-1} a_{l_i}' a_{l_i} - K I] c_i = 0 . \tag{7}$$

Formally, to solve for c_i, we should determine the largest root of the matrix $(a' a)^{-1} a_{l_i}' a_{l_i} c_i$, substitute this value for K in (7) and solve for all but one of the coordinates of c_i as proportions of that one coordinate. The c_i vector would then be normalyzed so that $c_i' c_i = 1$.

The foregoing outlines the formal statement and solution of the problem. However, since we do not in general know which tests will have significant loadings for any specified factor in the simple structure matrix, the matrix a_{l_i} is not available.

We therefore proceed in a somewhat different manner. First we premultiply both sides of (6) by a_{l_i} thus:

$$a_{l_i} (a' a)^{-1} a_{l_i}' a_{l_i} c_i = K a_{l_i} c_i . \tag{8}$$

If we let b_{l_i} be the vector of significant loadings for the ith column of b, then (8) becomes

$$a_{l_i}(a'a)^{-1}a_{l_i}'b_{l_i}=K\,b_{l_i} \tag{9}$$

or

$$[a_{l_i}(a'a)^{-1}a_{l_i}'-K\,I]\,b_{l_i}=K\,b_{l_i}. \tag{10}$$

The formal solution for b_{l_i} in (10) is well known. However, as yet we have not considered means for selecting a_{l_i}.

II. Approximation Procedures

Suppose first that instead of the matrix $a_{l_i}(a'a)^{-1}a_{l_i}'$ we consider the complete matrix

$$a(a'a)^{-1}a'=A\,. \tag{11}$$

As a first approximation we make the following assumptions.

1. Each factor has significant loadings for two and only two tests.
2. Each test has significant loadings in no more than one factor.
3. The number of factors in the simple structure factor matrix b is equal to m, the number of columns in the arbitrary matrix a.

Our first step is to select two tests for each factor in such a way that no test appears in more than one factor.

We begin by assuming that each test has a significant loading in some one of the factors. Considering first test 1, we ask which test when combined with test 1 will give the highest value for K in (10). From the matrix A in (11) we can actually construct all second-order matrices involving test 1. These matrices would be of the form

$$a_{l_i}(a'a)^{-1}a_{l_i}'=\begin{Vmatrix} a_1'(a'a)^{-1}a_1 & a_1'(a'a)^{-1}a_j \\ a_j'(a'a)^{-1}a_1 & a_j'(a'a)^{-1}a_j \end{Vmatrix} \tag{12}$$

where the diagonal elements of the right side of (12) are respectively the 1st and the jth diagonal elements of A and where the non-diagonal elements are taken from the 1st row and jth column of A (or the jth row and 1st column, since both (11) and (12) are symmetrical).

For simplicity we rewrite (12)

$$\alpha=\begin{Vmatrix} \alpha_{11} & \alpha_{1j} \\ \alpha_{1j} & \alpha_{jj} \end{Vmatrix}. \tag{13}$$

It can readily be verified then that the largest root of (13) is

$$K_i=\frac{\alpha_{11}+\alpha_{jj}+\sqrt{(\alpha_{11}-\alpha_{jj})^2+4\,\alpha_{1j}^2}}{2}. \tag{14}$$

It would be possible to let j take in turn all values of j except 1, select from this series the test corresponding to the largest K, and pair this test with test 1.

Similarly we could pass to test 2 and find which test when paired with it would give the largest K. Likewise we could determine the test to be paired with each of the other tests. However, if we have n tests this would require $n(n-1)$ determinations of K by means of equations analogous to (14). If we have, say, 40 tests, which is not unusual for factor analysis studies, this would mean 1560 separate determinations. Considering that this would constitute only the first step in our first set of approximations, the computational labor would be excessive and we therefore look for estimates of the K's which are much more readily obtained but which we can regard as sufficiently accurate.

An examination of (14) shows that the larger the value of a_{1j} the larger will be the value of K. Therefore, we find the largest non-diagonal entry in column 1 and pair its corresponding test with test 1. We assume that the root corresponding to these tests is not significantly less than the largest root in the series represented in (14) where j takes all values but 1. It is conceivable that a fictitious set of data could be devised in which this assumption is not verified. However, the assumption has been found empirically to hold in the case of a number of A matrices derived both from experimental and fictitious data.

Accordingly we underline the highest non-diagonal entry in each column of the A matrix. With each test we then pair the test corresponding to the largest non-diagonal entry in the column. This procedure gives us a certain number of duplicate pairs. For example, if test 5 has the largest non-diagonal entry in column 1, test 1 may have the largest non-diagonal entry in column 5. In general then we have less than n but more than $\frac{n}{2}$ non-identical pairs. However, we shall have a number of overlapping pairs. Thus we may have (1–3) and (3–5), in which case test 3 is common to both pairs. According to assumption 2 each test has significant loadings in no more than one factor. Our next step is therefore to select our pairs in such a way that no overlapping occurs.

First we calculate by means of equation (14) the K's corresponding to each of the non-identical pairs of tests. From this series of K's we select the pair with the highest K. We then examine the remaining pairs to see if any of them includes either of the tests in the selected pair. All of those which do are eliminated from further consid-

eration as are also their corresponding K's. Next we select the pair corresponding to the highest remaining K. All remaining pairs which include either test in this pair are eliminated from further consideration together with their corresponding K's.

Next we select the pair corresponding to the largest remaining K and eliminate from the remaining pairs and K's as before. This process is continued until the number of pairs selected is equal to the number of columns in the a matrix since, by assumption 3, the number of factors in b is equal to the number of columns in a.

In our next operation we begin with the pair of tests having the largest K and proceed by means of equation (9). We write down the matrix $a_{l_i}(a'\,a)^{-1}\,a_{l_i}'$ in which case a_{l_i} consists of the two rows of a corresponding to the pair of selected tests. This matrix can be copied directly from the elements of the A matrix given by (11). We then solve for $\delta\,b_{l_i}$ by means of the well known iteration method described by Hotelling (1), where δ is a constant such that the larger of the two values in b_{l_i} is unity. This method also gives us at the same time and independently the value of K. It does not give the absolute values for b_{l_i} but only values proportional to them.

Consider now equation (6). If we premultiply each side of a we have,

$$a(a'\,a)^{-1}\,a_{l_i}'\,a_{l_i}\,c_i = K\,a\,c_i\ ; \tag{15}$$

or, remembering that

$$\left.\begin{aligned} a_{l_i}\,c_i &= b_{l_i} \\ a\,c_i &= b_i \end{aligned}\right\}, \tag{16}$$

we have

$$a(a'\,a)^{-1}\,a_{l_i}'\,b_{l_i} = K\,b_i\ . \tag{17}$$

Since for the moment b_{l_i} is unknown we substitute for it the $b_{l_i}\,\delta$ in (17), thus,

$$a(a'\,a)^{-1}\,a_{l_i}'\,b_{l_i}\,\delta = \delta\,K\,b_i\ . \tag{18}$$

The right side of (18) gives values proportional to the complete vector of factor loadings corresponding to the factor whose significant loadings are assumed to be in the first pair of selected tests. It is readily obtained simply by multiplying the two columns of the A matrix which correspond to the selected pair of tests by $b_{l_i}\,\delta$.

Our next step is to determine which test when added to the two already selected will give the largest value of K in equation (9). First we define

$$b_{l_i}\,\mu = v_{l_i}\ , \tag{19}$$

where μ is a constant such that $v_{l_i}' v_{l_i} = 1$.
Instead of (18) we write

$$a(a' a)^{-1} a_{l_i} v_{l_i} = K v. \tag{20}$$

We indicate by v_j a value from the vector v corresponding to any test except the selected pair. It can then be shown* that the increment given to K by adding test j to the selected pair is given approximately by

$$\varepsilon = \frac{\alpha_{jj} + \sqrt{(K - \alpha_{1j}) + 4v_j^2}}{2}, \tag{21}$$

where α_{jj} is the diagonal entry in A corresponding to test j. This estimate of ε is always slightly under the true value; however, the underprediction is approximately the same for all values of v_j of about the same magnitude.

Furthermore, a large number of calculations indicate empirically that if, instead of using equation (21) as a criterion for selecting the next test, we select simply the test having the highest v_j, the final set of tests selected as having significant loadings for a given factor will be about or exactly the same. But since v is proportional to $b_i \delta$ in (18), we select the test having the highest value in $b_i \delta$ to go with the selected pair. We therefore augment the matrix α in (13) by bordering the lower and right-hand sides with the appropriate values from A, and copying the appropriate diagonal value from A into the lower hight-hand corner of α.

By applying Hotelling's interation method to the third-order α, we obtain a new $b_{l_i} \delta$ and K corresponding to the three tests. We multiply the three columns in A corresponding to the three tests by the new $b_{l_i} \delta$ to get a new $b_i \delta$. From the new $b_i \delta$ we find the highest value exclusive of those corresponding to the three selected tests and augment the α-matrix with the appropriate values from the A-matrix.

This process is continued for each factor until a sufficient number of significant loadings have been determined. No rigorous method has been developed for determining when the number of significant loadings is adequate. However, a simple criterion has been adopted which seems to give fairly good results.

We begin by specifying that when some function of K, and other relevant quantities, is less for the most recently added test than for the one immediately preceding then we have a sufficient number of loadings, excluding the one which caused the drop.

* Unpublished paper by the author.

We know that it is always possible to find a vector c_i which in the equation $a\,c_i = b_i$ will give at least $m-1$ zeros identically in b_i, where m is the number of columns in a. Therefore, K approaches unity as the number of significant loadings approaches $n-(m-1)$.

We cannot say, therefore, that so long as K does not decrease we are justified in adding additional tests, for irrespective of the experimental phenomena involved in the system, the mathematics guarantees that K will not decrease with the addition of another test.

Remembering that n is number of tests and that m is number of columns in a, and letting n_s be number of significant loadings, we look for some function

$$f = f(K, n, m, n_s) \tag{21}$$

such that when $n_s = n-(m-1)$, f will be less than for any smaller value of n_s. The most obvious function satisfying this condition is

$$f = K(n - m + 1 - n_s)^g, \tag{22}$$

where g is positive. When $n_s = n - m + 1$, f in (22) vanishes. For any smaller value of n_s, f is positive. The value of g has been determined purely empirically. It has been found in a large number of calculations that if $g = \frac{1}{2}$ and selections are made until f drops further selections do not materially change the b_i vector. Other values which have been tested are $g = \frac{2}{3}, \frac{3}{4}, 1$. Even for $g = \frac{2}{3}$ it was found that in many cases additional selections materially changed the b_i vector.

Undoubtedly it would be possible to develop a more logical and mathematically elegant criterion, and it is hoped that the problem will be attacked by competent mathematicians interested in the subject.

It will be remembered that the procedures outlined above do not give the absolute values in b_i but only values proportional to them. That is, instead of b_i we have $K\,\delta\,b_i$ from (18). It is necessary, therefore, to solve for δ at the same time we solve for c_i. To do this we go back to equation (6),

$$(a'\,a)^{-1}\,a_{l_i}'\,a_{l_i}\,c_i = K\,c_i\,, \tag{6}$$

from which we write

$$(a'\,a)^{-1}\,a_{l_i}\,b_{l_i}\,\delta = K\,\delta\,c_i = P. \tag{23}$$

Since by this time we have made the selections which determine a_{l_i}, we can solve for $P = K\,\delta\,c_i$ in (23). Since $c_i'\,c_i = 1$, we have from (12)

$$P'\,P = K^2\,\delta^2\,. \tag{24}$$

Therefore, to get b_i we simply multiply the right-hand side of

(18) by $\frac{1}{K\delta}$, so that

$$b_i = \frac{K\delta b_i}{\sqrt{P'P}}. \tag{25}$$

For c_i we have from (23)

$$c_i = \frac{P}{\sqrt{P'P}}. \tag{26}$$

The procedures outlined above are applied first, beginning with the pair of tests having the largest K. Next they are applied to the pair having the next largest K and so on. Sometimes it will be found that several pairs of K's will produce the same b_i vector. When this is the case it usually becomes evident before all of the selections for the duplicating factor have been made. As soon as we observe that the first r selections for any factor are the same as the first r selections for a previously completed factor, we know that the factor will be a duplicate and therefore pass on at once to the next factor.

III. Numerical Example

In order to clarify the foregoing procedures we shall carry thru a numerical example. It would be possible to illustrate the analysis on a small set of fictitious data. However, it is believed that the application of the method to actual experimental material will be more enlightening. We have selected for this purpose data from a study by Thurstone (2). This study was set up to throw more light on a factor previously identified as a "perceptual" factor. The simple structure factor matrix was obtained from the centroid matrix by graphical methods.

In applying our method of Thurstone's data, we begin with the centroid matrix by him. This we designate as a.
The steps are as follows:

1. Calculate the matrix $a'a$ given in Table 1.
2. Calculate $(a'a)^{-1}$ given in Table 2.
3. Calculate $a(a'a)^{-1}$ given in Table 3.
4. Calculate $a(a'a)^{-1}a' = A$ given in Table 4.
5. Underline the highest non-diagonal entry in each column of Table 4 and construct Table 5.

The first of each pair of numbers in column 1 of table 5 corresponds to a column of Table 4. The second corresponds to the highest non-diagonal entry in the column. Next we examine column 1 for du-

plicate pairs and eliminate the second of each duplicate pair. The remaining pairs are given in column 2.

By means of equation (14) the roots corresponding to each of the pairs in column 2 are calculated. For example, for the pair 1-8 we have:

.353 — diagonal entry in column 1 of Table 4;
.211 — largest non-diagonal entry in column 1;
.330 — diagonal entry in column 8 of Table 4.

We calculate

$$K = \frac{.353 + .330 + \sqrt{(.353 - .330) + 4(.211)^2}}{2} = .553.$$

Similarly we calculate a K value for each of the pairs in column 2. The K values are given in column 3.

We find the pair with the highest K in column 3. This is the pair (9-10) with a K of .925. The remaining pairs are examined to see whether any of the other pairs involve either 9 or 10. Since they do not, we then find the next highest K, namely, .813, associated with the pair (19-31). Neither 19 nor 31 is involved in any of the remaining pairs so we proceed to (4-11) with a K of .660. We find that 4 is involved in (21-4) and 11 in (7-11) so we eliminate both these pairs. We continue this procedure until 10 pairs have been selected. The ten selected pairs and their corresponding K's are given in order of size of K in columns 4 and 5 respectively.

6. Normally we should now begin with the pair with the largest K, namely (9-10). However, since it will be necessary for purposes of illustration to carry through the operations for only one factor, we shall select a pair which more typically illustrates the method. For this purpose we select the pair (3-22). First we construct the upper left-hand 2×2 square of Table 6 by copying from Table 4 the diagonals from columns 3 and 22 respectively and the non-diagonal from column 3, row 22.

To this 2 by 2 table we apply the Hotelling iteration method and get the values .904 and 1.000 corresponding to 3 and 22 respectively. At the same time we get the value $K = .639$.

7. Next we consider the equation $n_i = n - m + 1 - n_s$, where

$n = 31$, the number of rows in the a matrix;

$m = 10$ the number of columns in the a matrix;

$n_s = 2$ the number of significant loadings.

Therefore $n_i = 20$. We write 20 in row 1, column *a* of Table 7, and fill in the remainder of the column with 19, 18, 17, etc., so that the number of entries will be in excess of the maximum number of final significant loadings expected for any factor. The entries in column 6 of Table 7 are the square roots of those in column *a*.

We write the value $K = .639$ in column 4, row 1, of Table 7, since the pair (3-22) comes in the 4th position in column 4 of Table 5. Normally we would begin with the pair (9-10) and write its corresponding K in column 1.

8. Multiply the entry in row 1, column *a*, of Table 7, by the entry in row 1, column 4, which gives $4.472 \times .639 = 2.858$. We enter 2.858 in row 1, column 4 of Table 8.

9. Write the values .904 and 1.000 in column 1*a*, rows 3 and 22 respectively of Table 9. Then perform the vector multiplication of rows 3 and 22 of Table 4 by the values in column 1*a* of Table 12, and enter the values in column 16 of Table 9.

9*a*. Find the largest value in column 1*b* of Table 9 exclusive of those in rows 3 and 22. This is .478 in row 12. Therefore, we copy into Table 6 the values from Table 4 corresponding to (3-12), (22-12) and (12-12). We then have the 3×3 matrix in the upper left-hand corner. To this we apply the Hotelling iteration method and get the vector .883, 1.000, and .799 together with the value $K = .852$.

9*b*. Enter the value $K = .852$ in row 2, column 4 of Table 7 and multiply by entry in row 2, column *b* of Table 7. This gives $4.359 \times .852 = 3.714$. We enter the value 3.714 in row 2, column 4 of Table 8. We observe that 3.714 is larger than the value immediately above it, namely 2.858, therefore we conclude that test 12 has a significant loading in Factor 4. We proceed to determine whether there is a 4th significant loading in this factor.

9*c*. Copy the values .883, .799 and 1.000 given by step 9*a* into the appropriate rows of column 2*a* of Table 9. Perform with these values the vector multiplication on rows 3, 12, and 22 of Table 4, and enter the results in column 2*b* of Table 9.

9*d*. The highest value exclusive of the three already selected in column 2*b* of Table 9 is .194 in row 6. We therefore copy the values from Table 4 corresponding to (3-6), (22-6), (12-6), and (6-6) into Table 6, which gives the 4×4 square matrix in the upper left-hand corner.

Going through each of the successive steps as before, we get for the 3rd entry in column 4 of Table 8 the value 3.734. This is greater than the preceding value 3.714. We conclude that there are at least four significant loadings in Factor 4 and proceed to test for a fifth loading following the previous routine.

The next value in column 4 of Table 8 is 3.822 which is greater than the preceding value. We therefore test for a 6th factor. We get 3.768 in column 4 of Table 8. This is less than the preceding value 3.822. We therefore conclude that Factor 4 has only 5 significant loadings. These correspond to the tests 3, 6, 12, 22, 27.

10. Column 4*b* of Table 9 gives values proportional to the factor loadings in Factor 4. To get the proportionality factor we perform the vector multiplication of the entries in column 4*b* of Table 9 on the corresponding rows of Table 3. This gives column 4 of Table 10. We get the sum of the squares of the entries in this column which we enter in row *a* below the column. This value is 2.0511. We calculate the reciprocal of the square root of this value, namely, $\frac{1}{\sqrt{2.0511}} =$ .6982 and enter it immediately below in row *b*.

11. To get the actual values of the factor loadings for Factor 4 we multiply column 4*b* of Table 9 by the value .6982 in column 4, row *b* of Table 10 and enter the result in column 4 of Table 11.

12. Column 4 of Table 10 is also multiplied by the value .6982 and the result entered in column 4 of Table 12.

The foregoing procedure is applied to each of the pairs in column 4 of Table 5 and the entries are made in the appropriate tables. The same tables are used for all of the factors except Tables 6 and 9. Each factor requires a Table 6 and 9 of its own.

It will be noted that while the *a* matrix has 10 columns, the simple structure factor matrix given by Table 11 has only 9 factors. The reason for this is that the pairs (14-15) and (2-26) led to the same simple structure factors.

The tables of principal interest are 11 and 12. Table 12 is the matrix which transforms the centroid factor matrix *a* into the simple structure factor matrix of Table 11. A comparison of the simple structure matrix obtained by Thurstone and the one in Table 11 will be reserved for a future paper. An examination of the table indicates, however, that with several exceptions the factors are easily identifiable as those isolated by Thurstone.

In this analysis we failed to get any factor resembling Thurstone's *P* factor. However, our Factor 8 in Table 11 seems to call for speed in visual perception of small details. Our Factor 9 is very highly correlated with Factor 4 which is obviously the Number factor, although earlier approximations suggested that it might be the *I* or Induction factor.

TABLE 1 — $a'a$

	1	2	3	4	5	6	7	8	9	10
1	7.416	.403	−.486	−.056	.060	−.227	−.036	.063	.013	−.002
2	.403	3.099	.377	.086	−.032	.324	.065	.320	.042	.098
3	−.486	.377	1.749	−.118	−.230	.114	−.142	.028	−.131	.000
4	−.056	.086	−.118	1.222	.082	.010	−.170	−.077	−.083	.008
5	.060	−.032	−.230	.082	.945	.060	−.026	−.122	.073	.089
6	−.227	.324	.114	.010	.060	.990	−.154	.097	.044	−.036
7	−.036	.065	−.142	−.170	−.026	−.154	.722	−.003	.127	.012
8	.063	.320	.028	−.077	−.122	.097	−.003	.690	−.083	.009
9	.013	.042	−.131	−.083	.073	.044	.127	−.083	.505	.034
10	−.002	.098	.000	.008	.089	−.036	.012	.009	.034	.359
	7.148	4.782	1.161	.904	.899	1.222	.395	.922	.541	.571

TABLE 2 — $(a'a)^{-1}$

	1	2	3	4	5	6	7	8	9	10
1	.1415	−.0295	.0462	.0170	−.0041	.0426	.0313	−.0058	.0006	.0128
2	−.0295	.3742	−.0942	−.0603	−.0057	−.1148	−.0817	−.1644	−.0538	−.0996
3	.0462	−.0942	.6553	.0966	.1525	−.0398	.1328	.0763	.1554	−.0383
4	.0170	−.0603	.0966	.8834	−.0450	.0183	.2096	.1293	.1495	−.0141
5	−.0041	−.0057	.1525	−.0450	1.1688	−.1045	.0611	.2059	−.0878	−.2947
6	.0426	−.1148	−.0398	.0183	−.1045	1.1403	.2765	−.1508	−.1901	.1829
7	.0313	−.0817	.1328	.2096	.0611	.2765	1.5909	−.0130	−.3616	.0132
8	−.0058	−.1644	.0763	.1293	.2059	−.1508	−.0130	1.6348	.3173	−.0925
9	.0006	−.0538	.1554	.1495	−.0878	−.1901	−.3616	.3173	2.2352	−.1950
10	.0128	−.0996	−.0383	−.0141	−.2947	.1829	.0132	−.0925	−.1950	2.9236
	.2526	−.3298	1.1428	1.3843	1.0465	1.0606	1.8591	1.9371	1.9697	2.3983

TABLE 3 — $a(a'a)^{-1}$

	1	2	3	4	5	6	7	8	9	10
1	.0740	.0904	−.1108	.1354	−.0934	−.1280	−.3300	.2991	.4607	.2489
2	.0834	.0567	.0692	−.1484	.1423	.2134	−.1774	.0704	.3686	.1660
3	.0625	−.1313	−.1293	−.2336	−.1687	−.0463	−.0042	.2878	.2442	.1304
4	.0695	.0979	−.1816	−.1196	−.2003	.2684	.1272	−.3850	.1358	−.3735
5	.0795	−.1239	.2782	.0996	.0558	−.0063	.0325	−.1248	−.1703	.4817
6	.0826	.0567	.1621	−.1442	−.0846	−.1256	.3337	−.1644	−.3441	−.1446
7	.0705	.2027	−.0343	.0490	−.0190	.1151	.1681	−.1819	−.1921	.1999
8	.0524	.0810	−.2042	.1491	.1540	−.2455	−.2823	.2455	−.1245	.0111
9	.0665	−.1006	.0959	.1334	−.3140	.2393	−.2293	.3375	−.2822	−.3207
10	.0673	−.1087	.2288	−.1373	−.0087	.4048	−.2169	.2676	−.2357	−.1799
11	.0738	.1015	−.1673	.0644	−.2155	.3437	.1745	−.3897	−.0221	−.1572
12	.0759	−.0688	−.0802	−.2325	−.1814	−.1740	.0556	.1510	−.1336	.1705
13	.0598	.0680	.1561	.0886	−.2294	−.1341	−.1202	−.1115	.4627	−.3208
14	.0625	−.0859	−.1179	−.1181	.1038	−.2193	−.3228	−.1552	−.0171	−.3950
15	.0543	−.0690	.1500	−.3247	.4062	−.1047	−.2162	−.2945	.1975	−.2004
16	.0715	−.1170	.1606	.2797	−.1177	−.1284	−.1787	−.0058	.3552	.1539
17	.0772	−.1171	−.1385	.0754	.2374	.2910	.2369	−.1572	−.4758	.2653
18	.0805	−.1468	−.1067	−.0619	.1855	.1189	−.0965	.0309	−.1387	−.2270
19	.0803	−.1641	.1757	.3093	.1488	.1694	.5597	.3474	.2685	.2450
20	.0497	.1735	−.0866	−.1352	−.1010	.1074	−.1476	−.0324	.0096	.3858
21	.0694	−.1043	−.0900	.1000	−.0759	.2003	.1565	−.1040	.1763	−.3162
22	.0601	−.1155	−.1241	−.2251	−.2582	−.0649	.1857	.2578	.0721	.3535
23	.0566	−.0579	.1205	.2618	−.3002	−.1834	−.0785	−.3427	.2304	−.1677
24	.0543	.0545	.1129	−.0940	−.0808	.1286	.1720	−.2799	−.4307	.4801
25	.0563	−.1187	−.0711	.2318	−.2088	.0813	−.0327	−.2506	−.2218	.6167
26	.0504	−.0971	.1560	−.0515	.2194	.1743	−.1449	−.0646	.3520	.3259
27	.0785	.0728	.1038	−.1148	−.0226	−.3104	.0564	−.1404	−.2893	.0706
28	.0597	.1399	.1629	−.0791	.1916	−.1753	−.0355	.0704	−.0694	−.3732
29	.0601	.0893	−.1470	.1537	.2717	−.1313	−.0216	.0698	−.4067	−.3204
30	.0715	.1293	.0861	.0808	.2062	−.1519	−.0721	.2097	.2005	−.0410
31	.0634	−.0798	.2098	.2261	.2814	−.0332	.5940	.3587	.1259	−.5688
	2.0740	−.3923	.6390	.2181	−.0761	.4933	.1454	−.1810	.1059	.1989

TABLE 4 — $a(a'a)^{-1}\,a' = A$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
1	.353	.148	.102	−.015	−.038	−.133	.061	.211	.030	−.054	.003	.033	.102	−.002	−.092	.135	−.085	−.023	.050	.152	−.009	.068	.022	−.086	.041	.051	−.025	.011	.040	.172	−.066
2	.148	.287	.057	.079	.016	−.060	.076	−.009	−.031	.136	.053	−.027	.050	−.035	.143	.012	−.005	.029	.058	.162	.009	.007	−.102	.022	−.070	.207	−.050	.058	−.065	.125	−.034
3	.102	.057	.304	.046	−.058	.008	−.085	.006	.016	.013	−.036	.226	−.001	.070	.019	−.001	−.028	.076	.055	.033	.061	.304	−.053	−.061	−.021	.014	.007	−.057	−.090	−.023	.000
4	−.015	.079	.046	.367	−.147	.062	.106	−.090	−.023	−.024	.312	.001	.111	.055	.038	−.073	.030	.052	−.060	.071	.189	.023	.061	.002	−.044	−.021	−.034	−.003	−.018	−.040	−.004
5	−.038	.016	−.058	−.147	.288	.081	.022	−.035	.022	.077	−.080	.015	−.003	−.017	.085	.150	.096	.000	.106	−.009	−.057	−.018	.100	.170	.180	.130	.115	.009	−.034	.014	−.014
6	−.133	−.060	.008	.062	.081	.308	.094	−.083	.015	.022	.034	.135	.064	.002	.039	−.050	−.013	−.036	−.003	.002	−.017	.080	.059	.163	−.041	−.098	.228	.142	.022	.011	.122
7	.061	.076	−.085	.106	.022	.094	.232	.048	−.045	−.038	.162	−.021	.014	−.104	−.078	−.048	.075	−.060	.031	.158	−.016	−.035	−.029	.149	.041	−.017	.077	.071	.085	.095	−.005
8	.211	−.009	.006	−.090	−.035	−.083	.048	.330	.038	−.076	−.049	.036	−.050	.116	−.067	.040	.071	.083	−.041	.055	−.037	−.018	−.033	−.075	.060	−.064	.044	.055	.259	.139	−.027
9	.030	−.031	.016	−.023	.022	.015	−.045	.038	.538	.403	.024	.032	.061	.007	−.198	.065	−.038	.076	−.044	−.014	.041	−.012	.058	−.010	.035	−.098	−.034	.033	.045	−.032	.006
10	−.054	.136	.013	−.024	.077	.022	−.038	−.076	.403	.509	−.021	−.002	−.010	−.022	.033	−.022	.012	.106	−.026	.035	−.005	−.036	−.101	.062	−.064	.082	−.049	.078	−.039	−.012	−.005
11	.003	.053	−.036	.312	−.080	.034	.162	−.049	.024	−.021	.331	−.051	.072	−.014	−.068	−.036	.104	.031	−.002	.087	.172	−.032	.074	.056	.060	−.025	−.048	−.034	.028	−.032	−.016
12	.033	−.027	.226	.001	.015	.135	−.021	.036	.032	−.002	−.051	.260	−.017	.078	.000	−.020	−.001	.046	−.012	.036	−.006	.274	−.019	.048	.024	−.067	.141	.008	−.012	−.020	−.016
13	.102	.050	−.001	.111	−.003	.064	.014	−.050	.061	−.010	.072	−.017	.274	.031	.029	.150	−.208	−.078	−.018	.009	.058	−.034	.229	−.046	−.048	.007	.049	.085	−.075	.072	.045
14	−.002	−.035	.070	.055	−.017	.002	−.104	.116	.007	−.022	−.014	.078	.031	.312	.219	.052	.049	.188	−.166	−.080	.080	−.001	.105	−.096	.018	.001	.066	.036	.123	−.018	−.050
15	−.092	.143	.019	.038	.085	.039	−.078	−.067	−.198	.033	−.068	.000	.029	.219	.464	.013	.034	.135	−.088	−.040	.015	−.060	.006	−.002	−.094	.216	.070	.100	−.027	.030	−.015
16	.135	.012	−.001	−.073	.150	−.050	−.048	.040	.065	−.022	−.036	−.020	.150	.052	.013	.266	−.047	−.007	.098	−.043	.041	−.024	.238	−.033	.153	.094	.016	−.029	−.040	.048	.004
17	−.085	−.005	−.028	.030	.096	−.013	.075	.071	−.038	.012	.104	−.001	−.208	.049	.034	−.047	.369	.180	.119	−.008	.089	−.005	−.088	.101	.180	.060	−.018	−.078	.156	−.047	.026
18	−.023	.029	.076	.052	.000	−.036	−.060	.083	.076	.106	.031	.046	−.078	.188	.135	−.007	.180	.225	.000	−.053	.113	.020	−.034	−.055	.031	.051	−.029	−.008	.121	−.023	.033
19	.050	.058	.055	−.060	.106	−.003	.031	−.041	−.044	−.026	−.002	−.012	−.018	−.166	−.088	.098	.119	.000	.459	−.055	.095	.093	−.020	−.004	.045	.104	−.066	−.043	−.029	.079	.335
20	.152	.162	.033	.071	−.009	.002	.158	.055	−.014	.035	.087	.036	.009	−.080	−.040	−.043	−.008	−.053	−.055	.223	−.062	.052	−.092	.107	.017	.045	.032	.031	−.014	.084	−.160
21	−.009	.009	.061	.189	−.057	−.017	−.016	−.037	.041	−.005	.172	−.006	.058	.080	.015	.041	.089	.113	.095	−.062	.204	.028	.095	−.076	.031	.013	−.080	−.058	.013	−.046	.111
22	.068	.007	.304	.023	−.018	.080	−.035	−.018	−.012	−.036	−.032	.274	−.034	−.001	−.060	−.024	−.005	.020	.093	.052	.028	.365	−.060	.023	.028	−.035	.064	−.074	−.105	−.043	.004
23	.022	−.102	−.053	.061	.100	.059	−.029	−.033	.058	−.101	.074	−.019	.229	.105	.006	.238	−.088	−.034	−.020	−.092	.095	−.060	.368	−.014	.147	−.022	.075	−.016	−.036	−.029	−.004
24	−.086	.022	−.061	.002	.170	.163	.149	−.075	−.010	.062	.056	.048	−.046	−.096	−.002	−.033	.101	−.055	−.004	.107	−.076	.023	−.014	.258	.117	.026	.139	.025	−.029	−.019	−.090
25	.041	−.070	−.021	−.044	.180	−.041	.041	.060	.035	−.064	.060	.024	−.048	.018	−.094	.153	.180	.031	.045	.017	.031	.028	.147	.117	.335	.036	.026	−.153	.014	−.075	−.160
26	.051	.207	.014	−.021	.130	−.098	−.017	−.064	−.098	.082	−.025	−.067	.007	.001	.216	.094	.060	.051	.104	.045	.013	−.035	−.022	.026	.036	.271	−.065	−.024	−.109	.046	−.042
27	−.025	−.050	.007	−.034	.115	.228	.077	.044	−.034	−.049	−.048	.141	.049	.066	.070	.016	−.018	−.029	−.066	.032	−.080	.064	.075	.139	.026	−.065	.251	.133	.070	.057	.007
28	.011	.058	−.057	−.003	.009	.142	.071	.055	.033	.078	−.034	.008	.085	.036	.100	−.029	−.078	−.008	−.043	.031	−.058	−.074	−.016	.025	−.153	−.024	.133	.218	.109	.144	.120
29	.040	−.065	−.090	−.016	−.034	.022	.085	.259	.045	−.039	.028	−.012	−.075	.123	−.027	−.040	.156	.121	−.029	−.014	.013	−.105	−.036	−.029	.014	−.109	.070	.109	.326	.104	.098
30	.172	.125	−.023	−.040	.014	.011	.095	.139	−.032	−.012	−.032	−.020	.072	−.018	.030	.048	−.047	−.023	.079	.084	−.046	−.043	−.029	−.019	−.075	.046	.057	.144	.104	.203	.104
31	−.066	−.034	.000	−.004	−.014	.122	−.005	−.027	.006	−.005	−.016	−.016	.045	−.050	−.015	.004	.026	.033	.335	−.160	.111	.004	−.004	−.090	−.160	−.042	.007	.120	.098	.104	.495
	1.157	1.246	.903	1.011	1.166	1.159	1.016	.837	.966	.962	1.059	1.102	.924	1.003	.859	1.102	1.082	1.190	1.050	.761	.989	.841	.885	.772	.849	.767	1.169	.859	.893	1.068	.802

TABLE 5

Selection of Test Pairs Corresponding to Assumed Factors

1	2	3	4	5
1-8	1-8	.553	9-10	.925
2-26	2-26	.486	19-31	.813
3-22	3-22	.639	4-11	.660
4-11	4-11	.660	3-22	.639
5-25	5-25	.492	14-15	.619
6-27	6-27	.509	8-29	.586
7-11	7-11	.451	16-23	.561
8-29	8-29	.586	17-25	.533
9-10	9-10	.925	6-27	.509
10-9	12-22	.592	2-26	.486
11-4	13-23	.556		
12-22	14-15	.619		
13-23	16-23	.561		
14-15	17-25	.533		
15-14	18-14	.462		
16-23	19-31	.813		
17-18	20-2	.420		
18-14	21-4	.492		
19-31	26-15	.604		
20-2	28-30	.355		
21-4	30-1	.467		
22-3				
23-16				
24-5				
25-17				
26-15				
27-6				
28-30				
29-8				
30-1				
31-19				

α-Matrix Corresponding to Factor 4

TABLE 6

	3	22	12	6	27	24
3	.304	.304	.226	.008	.007	−.061
22	.304	.365	.204	.080	.064	.023
12	.226	.204	.260	.135	.141	.048
6	.008	.080	.135	.308	.228	.163
27	.007	.064	.141	.228	.251	.139
24	−.061	.023	.048	.163	.139	.258

TABLE 7

Successive Roots of the α Matrices

a	*b*	1	2	3	4	5	6	7	8	9	10
20	4.472	.925	.813	.660	.639	.619	.589	.561	.533	.509	.486
19	4.359	.948	.844	.776	.852	.718	.689	.725	.641	.633	.731
18	4.243	.958	.874	.823	.880	.781	.783	.783	.739	.729	.771
17	4.123		.890	.857	.926	.841	.823	.863	.798	.791	.837
16	4.000			.911	.942	.886	.858	.892	.843	.861	.877
15	3.873			.932		.916	.904	.923	.898	.949	.918
14	3.742					.951	.948	.937	.937	.960	.951
13	3.606					.960	.968		.950		.960
12	3.464										
11	3.317										
10	3.162										

TABLE 8

Criteria for Determining When All Tests with Significant Loadings Have Been Selected

1	2	3	4	5	6	7	8	9	10
4.137	3.636	2.952	2.858	2.768	2.634	2.509	2.384	2.276	2.173
4.123	3.679	3.383	3.714	3.130	3.003	3.160	2.794	2.759	3.186
4.065	3.708	3.492	3.734	3.314	3.322	3.322	3.136	3.093	3.271
	3.669	3.533	3.822	3.467	3.393	3.558	3.290	3.261	3.451
		3.644	3.768	3.544	3.432	3.568	3.372	3.444	3.508
		3.610		3.548	3.501	3.575	3.478	3.675	3.555
				3.559	3.547	3.506	3.506	3.592	3.559
				3.462	3.491		3.426		3.462

TABLE 9

Successive Approximations to Loadings in Factor 4

	1*a*	1*b*	2*a*	2*b*	3*a*	3*b*	4*a*	4*b*
1		.160		.185		.137		.103
2		.058		.036		.013		−.022
3	.904	.580	.883	.753	.858	.754	.821	.761
4		.065		.064		.085		.077
5		−.070		−.058		−.027		.041
6		.088		.194	.348	.306	.501	.465
7		−.112		−.128		−.093		−.041
8		−.012		.016		−.011		−.003
9		.003		.028		.033		.022
10		−.024		−.026		−.019		−.038
11		−.065		−.104		−.094		−.111
12		.478	.799	.681	.829	.730	.887	.822
13		−.035		−.048		−.027		.005
14		.062		.123		.124		.157
15		−.043		−.044		−.030		.007
16		−.024		−.040		−.059		−.060
17		−.030		−.031		−.035		−.043
18		.089		.124		.112		.092
19		.143		.132		.129		.096
20		.082		.110		.112		.126
21		.084		.078		.070		.027
22	1.000	.639	1.000	.852	1.000	.880	1.000	.926
23		−.108		−.121		−.100		−.056
24		−.032		.008		.067		.161
25		.009		.028		.016		.024
26		−.023		−.076		−.113		−.162
27		.070		.184		.266	.456	.424
28		−.125		−.118		−.067		.019
29		−.186		−.194		−.185		−.147
30		−.063		−.079		−.075		−.048
31		.004		−.009		.033		.055
		1.663		2.520		2.932		3.679

TABLE 10

Columns Proportional to Transformation Vectors — c_i

	1	2	3	4	5	6	7	8	9
1	.1316	.1936	.2717	.2559	.3172	.3754	.3070	.3813	.3188
2	−.2057	−.3186	.2082	−.2226	−.3937	.6687	−.2887	−.5802	−.2028
3	.3171	.3039	−.2562	−.1728	.3683	−.2924	.4792	.5748	−.0242
4	.0007	.5923	.2148	−.7477	−.5459	.2714	.9111	.8973	−.7998
5	−.3224	.4832	−.7517	−.6103	.9182	.5122	−.8649	−.0938	−.6439
6	.6306	.3043	.5845	−.4619	.4583	−.4069	−.2592	.3214	−.4918
7	−.4389	1.2826	.3400	.4244	−.6440	−.7882	−.5772	.3688	.5628
8	.5961	.6109	−1.1374	.4815	−.3194	.7036	−.3547	−.6000	.2913
9	−.5100	.2891	.5343	−.1504	.1667	.2983	.6397	−.1044	−.4701
10	−.4946	−.3469	−.9136	.5716	.0079	.2830	.3859	1.8516	.8357
	−.2955	3.3944	−.9054	−.6323	.3336	1.6251	.3782	3.0168	−.6240
a	1.7145	3.1304	3.6651	2.0511	2.2870	2.4605	3.0697	5.6650	2.7605
b	.7637	.5652	.5224	.6982	.6613	.6375	.5707	.4202	.6019

TABLE 11

The Simple Structure Matrix — *a c*

		1	2	3	4	5	6	7	8	9
Abstract Classification	1	–.02	–.03	.04	.07	–.05	.61	.21	.04	.02
" $R/(R+0)$	2	.08	.01	.07	–.02	.36	.34	–.03	.02	–.02
Additions	3	.02	.04	–.01	.53	.07	.00	–.03	–.05	.43
Anagrams	4	–.04	.01	.48	.05	.04	.00	–.04	–.10	.03
Areas	5	.07	.06	–.10	.03	.25	–.04	.29	.39	.12
Arithmetical Reasoning	6	.03	.06	.10	.32	–.03	–.06	.01	.02	.38
Completion	7	–.06	.03	.17	–.03	–.08	.36	–.01	.06	.03
Concrete Association	8	–.03	–.03	–.10	.00	–.03	.58	.06	–.01	–.03
Designs	9	.71	–.02	.04	.02	–.01	.02	.28	.01	.01
" $R/(R+0)$	10	.68	–.02	–.06	–.03	.36	–.03	.04	.00	.00
Disarranged Words	11	.00	.04	.46	–.08	–.03	.07	.05	.03	–.07
Division	12	.02	–.02	–.04	.57	.00	.03	.00	.00	.53
Flags	13	.04	–.01	.26	.00	–.04	.09	.33	.02	.00
Highest Number	14	–.01	–.10	.07	.11	.30	.04	.10	–.01	.08
" $R/(R+W)$	15	–.13	–.05	–.01	.00	.63	–.06	–.07	.03	.02
Identical Forms	16	.03	.06	.06	–.04	.09	.08	.49	.30	–.03
Identical Numbers	18	.14	.08	.04	.06	.36	.04	–.02	.05	.03
Initials	19	.05	.49	–.01	.07	.01	.03	.08	.28	.06
Inventive Opposites	20	.01	–.13	.07	.09	.02	.39	–.04	–.02	.10
Letter *A*	21	.03	.17	.29	.02	.10	–.09	.10	.07	–.02
Multiplication	22	–.04	.06	–.03	.65	–.07	–.05	–.04	.01	.57
Pursuit	23	–.03	–.01	.27	–.04	–.06	–.13	.53	.24	–.01
Reasoning	24	.04	–.05	.02	.11	.06	–.01	.03	.19	.21
Scattered *X*'s	25	–.02	–.02	.05	.02	.02	–.02	.35	.39	.06
" $R/(R+0)$	26	–.01	.05	–.03	–.11	.45	.05	.08	.21	–.09
Tabular Completion	27	–.06	–.05	–.02	.30	.01	.14	.10	.07	.35
Verbal Analogies	28	.08	.02	.00	.01	.09	.29	–.04	–.11	.05
Verbal Enumerations	29	.01	.07	–.01	–.10	.01	.42	–.05	–.03	–.09
Word Grouping	30	–.03	.08	–.02	–.03	.06	.51	.04	.00	–.02
Word Number	31	.00	.49	.04	.04	–.04	.00	–.06	.01	.02
		1.44	1.45	2.12	2.56	3.12	3.63	2.66	2.35	2.73

TABLE 12

The Transformation Matrix — c

	1	2	3	4	5	6	7	8	9
1	.100	.109	.142	.179	.210	.239	.175	.160	.192
2	−.157	−.180	.109	−.155	−.260	.426	−.165	−.244	−.122
3	.242	.172	−.134	−.121	.244	−.186	.274	.242	−.015
4	.001	.335	.112	−.522	−.361	.173	.520	.377	−.481
5	−.246	.273	−.393	−.426	.607	.327	−.494	−.039	−.388
6	.482	.172	.305	−.323	.303	−.259	−.148	.135	−.296
7	−.335	.725	.178	.296	−.426	−.502	−.329	.155	.339
8	.455	.345	−.594	.336	−.211	.449	−.202	−.252	.175
9	−.390	.163	.279	−.105	.110	.190	.365	−.044	−.283
10	−.378	−.196	−.477	.399	.005	.180	.220	.778	.503
	−.226	1.918	−.473	−.442	.221	1.037	.216	1.268	−.376

REFERENCES

1. Hotelling, H.. Analysis of a complex of statistical variables into principle components. Baltimore: Warwick and York, 1933.
2. Thurstone, L. L. The perceptual factor. *Psychometrika*, 1938, **3**, 1-17.
3. Thurstone, L. L. A new rotational method in factor analysis. *Psychometrika*, 1938, **3**, 199-218.
4. Tucker, L. R. A Rotational method based upon the mean principal axis of a subgroup of tests. *Psychological Bulletin*, 1940, **37**, 578.

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THE SOLUTION OF SIMULTANEOUS EQUATIONS

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This paper is an attempt to integrate the various methods which have been developed for the numerical solution of simultaneous linear equations. It is demonstrated that many of the common methods, including the Doolittle Method, are variations of the method of "single division." The most useful variation of this method, in case symmetry is present, appears to be the Abbreviated Doolittle method. The method of multiplication and subtraction likewise can be abbreviated in various ways of which the most satisfactory form appears to be the new Compact method. These methods are then applied to such problems as the solution of related equations, the solution of groups of equations, and the evaluation of the inverse of a matrix.

Introduction

During recent years various articles have appeared, (1) (2) (3) (4), on the general topic of the solution of normal equations. At first thought it appears that articles of this type are not needed since the theory of determinants gives the desired solution, but further invesvestigation reveals that classical determinantal methods, excellent for theoretical purposes, are not the most efficient tools for determining numerical solutions when the number of variables is large. The improvement of the modern calculating machine (in particular the introduction of automatic division and automatic multiplication) has made available methods which, though algebraically simple, were previously numerically cumbersome. It appears now that the new numerical methods have been developed sufficiently so that the presentation of a rather thorough treatment of them, and of their relations to each other, is wise. From such an analysis we are able to arrive at new and more compact methods. It should be specified also that we are not at present investigating iterative nor determinantal methods, though the methods outlined in this paper can be used in evaluating determinants.

In many of the problems in which the approximate solution of a large number of equations is desired, the simultaneous equations are symmetric with respect to the coefficients of the unknowns (i.e., the matrix of the coefficients of the unknowns is symmetric). For example, if the equations are

$$\begin{aligned} a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 &= a_{51} \\ a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4 &= a_{52} \\ a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + a_{43}x_4 &= a_{53} \\ a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + a_{44}x_4 &= a_{54} \end{aligned} \tag{1}$$

then, frequently $a_{ij} = a_{ji}$. This is always true, of course, if the equations are normal equations or their equivalents. This symmetry, if present, simplifies the resulting techniques. In the present article we give the techniques for the general situation and also the specialized techniques in case symmetry is present.

The methods outlined apply to the solution of n equations in n unknowns. In the interest of brevity, however, we use but four equations in four unknowns, equations (1), to present the theory. The solutions are not more complicated, just longer, when more than four variables are involved.

Each of the methods presented is illustrated by application to the equations

$$\begin{aligned} x_1 + .4x_2 + .5x_3 + .6x_4 &= .2 \\ .4x_1 + 1.0x_2 + .3x_3 + .4x_4 &= .4 \\ .5x_1 + .3x_2 + 1.0x_3 + .2x_4 &= .6 \\ .6x_1 + .4x_2 + .2x_3 + 1.0x_4 &= .8 \end{aligned} \tag{2}$$

where the a's are exact values. The solutions in this case are then carried to five significant figures or to four decimal places. All solutions are indicated symbolically in the tables where no x's appear explicitly, but the first column presents the coefficients of the x_1, the second of x_2, etc. The last column is the conventional check column giving the sum of the entries in the row.

Method One. The Method of Division

The first method described, which is essentially that given in Huntington's article in the "Handbook of Mathematical Statistics" (5, p. 67), we might call the method of division. Each equation is divided by its leading coefficient and some equation, say the first, is subtracted in turn from each of the $n - 1$ others giving a new set of $n - 1$ equations in $n - 1$ unknowns. The process is repeated until one equation in one unknown is obtained. The back solution is then readily obtained by substitution.

TABLE 1. METHOD OF DIVISION

General						Illustration					
x_1	x_2	x_3	x_4		check	x_1	x_2	x_3	x_4		check
a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	1.0000	.4000	.5000	.60000	.2000	2.7000
a_{12}	a_{22}	a_{32}	a_{42}	a_{52}	a_{62}	.4000	1.0000	.3000	.40000	.4000	2.5000
a_{13}	a_{23}	a_{33}	a_{43}	a_{53}	a_{63}	.5000	.3000	1.0000	.20000	.6000	2.6000
a_{14}	a_{24}	a_{34}	a_{44}	a_{54}	a_{64}	.6000	.4000	.2000	1.00000	.8000	3.0000
1	b_{21}	b_{31}	b_{41}	b_{51}	b_{61}	1.0000	.4000	.5000	.6000	.2000	2.7000
1	b_{22}	b_{32}	b_{42}	b_{52}	b_{62}	1.0000	2.5000	.7500	1.0000	1.0000	6.2500
1	b_{23}	b_{33}	b_{43}	b_{53}	b_{63}	1.0000	.6000	2.0000	.4000	1.2000	5.2000
1	b_{24}	b_{34}	b_{44}	b_{54}	b_{64}	1.0000	.6667	.3333	1.6667	1.3333	5.0000
	$a'_{22.1}$	$a'_{32.1}$	$a'_{42.1}$	$a'_{52.1}$	$a'_{62.1}$		2.1000	.2500	.4000	.8000	3.5500
	$a'_{23.1}$	$a'_{33.1}$	$a'_{43.1}$	$a'_{53.1}$	$a'_{63.1}$		.2000	1.5000	−.2000	1.0000	2.5000
	$a'_{24.1}$	$a'_{34.1}$	$a'_{44.1}$	$a'_{54.1}$	$a'_{64.1}$		.2667	−.1667	1.0667	1.1333	2.3000
	1	$b'_{32.1}$	$b'_{42.1}$	$b'_{52.1}$	$b'_{62.1}$		1.0000	.1190	.1905	.3810	1.6905
	1	$b'_{33.1}$	$b'_{43.1}$	$b'_{53.1}$	$b'_{63.1}$		1.0000	7.5000	−1.0000	5.0000	12.5000
	1	$b'_{34.1}$	$b'_{44.1}$	$b'_{54.1}$	$b'_{64.1}$		1.0000	−.6250	3.9996	4.2493	8.6239
		$a'_{33.12}$	$a'_{43.12}$	$a'_{53.12}$	$a'_{63.12}$			7.3810	−1.1905	4.6190	10.8095
		$a'_{34.12}$	$a'_{44.12}$	$a'_{54.12}$	$a'_{64.12}$			−.7440	3.8091	3.8683	6.9334
		1	$b'_{43.12}$	$b'_{53.12}$	$b'_{63.12}$			1.0000	−.1613	.6258	1.4645
		1	$b'_{44.12}$	$b'_{54.12}$	$b'_{64.12}$			1.000	−5.1198	−5.1993	−9.3191
			$a'_{44.123}$	$a'_{54.123}$	$a'_{64.123}$				−4.9585	−5.8251	−10.7836
			1	$b'_{54.123}$	$b'_{64.123}$				1.0000	1.1748	2.1748
		1		$b'_{53.124}$	$b'_{63.124}$			1.0000		.8153	1.8153
	1			$b'_{52.134}$	$b'_{62.134}$		1.0000			.0602	1.0602
1				$b'_{51.234}$	$b'_{61.234}$	1.0000				−.9366	.0634

The solution of (1) is given at the left of Table 1 while the four-decimal-place approximation to the solution of (2) is given at the right. If we refer to each main subdivision of the table as a "matrix," we may say that the first matrix denotes the equations. The second matrix denotes the equations resulting when each has been divided by its leading coefficient. The third matrix results from subtracting the first equation from the others. Thus $a'_{22.1} = b_{22} - b_{21}$, and in the illustration $2.1000 = 2.5000 - .4000$. This process is then continued until one equation in one unknown results and $x_4 = b'_{54.123}$. Now since $x_3 + b'_{43.12}\, x_4 = b'_{53.12}$, it follows that $x_3 = b'_{53.12} - b'_{43.12}\, b'_{54.123}$, and the value $x_3 = b'_{53.124}$ can be obtained from the entries in the first row of the sixth matrix and the x_4 previously obtained. Thus from $b'_{53.12}$ we subtract the product of $b'_{43.12}$ and the entry to the right of the "one" in the last matrix, $b'_{54.123}$. This result may be checked by using the second row of matrix 6 with the resulting

$$b'_{53.124} = b'_{54.12} - b'_{44.12}\, b'_{54.123}\,.$$

Similarly, using the first row of matrix 4

$$b'_{52.134} = b'_{52.1} - b'_{42.1}\, b'_{54.123} - b'_{32.1}\, b'_{53.124}\,,$$

etc. This process is carried out easily from the entries of the table once the "ritual" is learned.

The check column, column 6, is handled just as column 5, though column 5 is not involved in the back solution. Thus $b'_{63.124} = b'_{63.12} - b'_{43.12}\, b'_{64.123}$. The check results from the fact that, for every row in every matrix, the entry in column 6 is the sum of the entries in the columns to the left.

This method is the least satisfactory (aside from its theoretical simplicity) of the various methods presented. The symmetry of the equations, if originally present, is lost with the first set of divisions. Many divisions are demanded and the method utilizes $n(n+1) + n - 1 = (n+1)^2 - 2$ rows. In general the method will give approximations since divisions are involved.

Brolyer (6) and Chauncey (7) have shown how this method may be simplified by the elimination of considerable recording and some computing. The even-numbered matrices in Table 1 are replaced by a single row which is placed directly under the equation from which it was obtained. The processes of division and subtraction are then performed in one operation with the use of (a/b) − c techniques. The number of rows in the forward and back solution is reduced to $\frac{(n+1)(n+2)}{2} + n - 2 = \frac{n(n+5)}{2} - 1$. If $a_{11} = 1$, the first division row is not needed and we have $\frac{n(n+5)}{2} - 2$ rows.

Method Two. The Method of Single Division

The method of single division is characterized by the leading element in the case of but one of the n equations. The resulting equation is then combined with each of the $n-1$ other equations in turn to give $n-1$ equations in $n-1$ unknowns. The new equation can be written at the bottom of the n equations. There is no great loss in generality and a somewhat simpler technique results if the variables are eliminated in 1, 2, 3, 4, order, (divide the top row of each matrix by its leading coefficient), though a somewhat more complicated technique can be worked out if no such restriction is made. It is necessary that this leading coefficient be different from 0.

The method is derived by the elimination of x_1 from each of the last three equations of (1),

$$a_{1j}x_1 + a_{2j}x_2 + a_{3j}x_3 + a_{4j}x_4 = a_{5j}\,, \quad j = 2, 3, 4 \tag{3}$$

and from the first equation of (1) divided by its leading coefficient,

$$x_1 + b_{21}x_2 + b_{31}x_3 + b_{41}x_4 = b_{51}\,. \tag{4}$$

We multiply (4) by a_{1j} and subtract from (3) to get

$$(a_{2j} - a_{1j}\,b_{21})\,x_2 + (a_{3j} - a_{1j}\,b_{31})\,x_3 + (a_{4j} - a_{1j}\,b_{41})\,x_4 = a_{5j} - a_{1j}\,b_{51}\,,$$

which can be written

$$a_{2j.1}\,x_2 + a_{3j.1}\,x_3 + a_{4j.1}\,x_4 = a_{5j.1}\,, \qquad j = 2, 3, 4\,. \tag{5}$$

The process is repeated to eliminate x_2, x_3, etc., in turn. Thus

$$a_{3j.12}\,x_3 + a_{4j.12}\,x_4 = a_{5j.12}\,,$$

with $a_{ij.12} = a_{ij.1} - a_{2j.1}\,b_{i2.1}$.

The forward technique is simple. Divide the first row in any matrix by its leading coefficient to form the last row in the matrix. Take any element in the matrix and subtract the product of its lead-

TABLE 2. METHOD OF SINGLE DIVISION

General						Illustration					
x_1	x_2	x_3	x_4		check	x_1	x_2	x_3	x_4		check
a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	1.0000	.4000	.5000	.6000	.2000	2.7000
a_{12}	a_{22}	a_{32}	a_{42}	a_{52}	a_{62}	.4000	1.0000	.3000	.4000	.4000	2.5000
a_{13}	a_{23}	a_{33}	a_{43}	a_{53}	a_{63}	.5000	.3000	1.0000	.2000	.6000	2.6000
a_{14}	a_{24}	a_{34}	a_{44}	a_{54}	a_{64}	.6000	.4000	.2000	1.0000	.8000	3.0000
1	b_{21}	b_{31}	b_{41}	b_{51}	b_{61}	1.0000	.4000	.5000	.6000	.2000	2.7000
	$a_{22.1}$	$a_{32.1}$	$a_{42.1}$	$a_{52.1}$	$b_{62.1}$		.8400	.1000	.1600	.3200	1.4200
	$a_{23.1}$	$a_{33.1}$	$a_{43.1}$	$a_{53.1}$	$a_{63.1}$		.1000	.7500	−.1000	.5000	1.2500
	$a_{24.1}$	$a_{34.1}$	$a_{44.1}$	$a_{54.1}$	$a_{64.1}$		.1600	−.1000	.6400	.6800	1.3800
	1	$b_{32.1}$	$b_{42.1}$	$b_{52.1}$	$b_{62.1}$		1.0000	.1190	.1905	.3810	1.6905
		$a_{33.12}$	$a_{43.12}$	$a_{53.12}$	$a_{63.12}$			.7381	−.1190	.4619	1.0810
		$a_{34.12}$	$a_{44.12}$	$a_{54.12}$	$a_{64.12}$			−.1190	.6095	.6190	1.1095
		1	$b_{43.12}$	$b_{53.12}$	$b_{63.12}$			1.0000	−.1612	.6258	1.4646
			$a_{44.123}$	$a_{54.123}$	$a_{64.123}$				.5903	.6935	1.2838
			1	$b_{54.123}$	$b_{64.123}$				1.0000	1.1748	2.1748
		1		$b_{53.124}$	$b_{63.124}$			1.0000		.8152	1.8152
	1			$b_{52.134}$	$b_{62.134}$		1.0000			.0602	1.0602
1				$b_{51.234}$	$b_{61.234}$	1.0000				−.9366	.0634

ing row element and its bottom columnar element and the result is the proper entry for the corresponding place in the next matrix. The solution of (1), with the added check column, is shown in Table 2. This method, essentially, has been given by Deming (8), who did not restrict the order of elimination. It has also been given, essentially, by Aitken (9) who calls it the "Method of pivotal condensation," though Aitken placed the divided row at the top of the matrix.

The back solution is obtained by subtraction. Thus $b_{53.124}$ is obtained by substituting $x_4 = b_{54.123}$ in the equation $x_3 + b_{43.12}\, x_4 = b_{53.12}$ so that $x_3 = b_{53.12} - b_{43.12}\, b_{54.123}$. This is easily done since it is only necessary to record $b_{53.12}$ and subtract from it the product of the term before it ($b_{43.12}$) and the term associated with the "one" below it ($b_{53.124}$). Similarly the computation of $b_{52.134} = b_{52.1} - b_{42.1}\, b_{54.123} - b_{32.1}\, b_{53.124}$ can be reduced to a simple ritual.

This method is much preferable to the method of division in that fewer divisions and subtractions are required and there is less recording. Also this solution demands but $\frac{n^2 + 5n - 2}{2}$ rows for the forward and back solutions. Furthermore this solution retains symmetry, if originally present, for if $a_{ij} = a_{ji}$, then

$$a_{ij.1} = a_{ij} - a_{1j}\, b_{i1} = a_{ij} - \frac{a_{1j}\, a_{i1}}{a_{11}} = a_{ji.1},$$

$$a_{ij.12} = a_{ji.12}, \text{ etc.}$$

This symmetry has not been used (though it could be used for checking) in the foregoing method so that this method is applicable to non-symmetric equations.

Method Three. Method of Single Division—Symmetric

This method is essentially a special case of the method above though some adjustments are needed because of deleted entries. It is not necessary to record both $a_{ij.}$ and $a_{ji.}$ since they are equal if $a_{ij} = a_{ji}$. We hence omit all the entries below the main diagonal for each matrix. For example, in Table 2 we omit all the entries to the left of and below the main diagonal. A dash (—) is used to indicate the omission, rather than the vanishing, of these entries.

However, we have been using the entries at the left of the table in computing the entries in the next matrix and so we still have to find the equivalent of these entries. Now a duplicate of the leading term for each row is obtained by going along the row to the main

diagonal and then taking the heading of that column, since $a_{ij} = a_{ji}$. In this manner the subtracted product is obtained from the first and last rows of each matrix. The entry from the last row is the entry at the foot of the column, while the entry from the top row is obtained by moving along the row which contains the element to the main diagonal of the matrix and then taking the heading of that column. Thus

$$a_{ij.1} = a_{ij} - b_{i1}\, a_{j1},$$

and in Table 2

$$a_{34.1} = (.2000) - (.5000)(.6000) = -.1000.$$

Similarly,

$$a_{34.12} = (-.1000) - (.1000)(.1905) = -.1190.$$

The back solution is similar to that of method 2. The technique is easily learned with a little practice. The solution of (1) is presented in Table 3.

TABLE 3. METHOD OF SINGLE DIVISION—SYMMETRIC

General						Illustration					
x_1	x_2	x_3	x_4		check	x_1	x_2	x_3	x_4		check
a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	1.0000	.4000	.5000	.6000	.2000	2.7000
—	a_{22}	a_{32}	a_{42}	a_{52}	a_{62}	—	1.0000	.3000	.4000	.4000	2.5000
—	—	a_{33}	a_{43}	a_{53}	a_{63}	—	—	1.0000	.2000	.6000	2.6000
—	—	—	a_{44}	a_{54}	a_{64}	—	—	—	1.0000	.8000	3.0000
1	b_{21}	b_{31}	b_{41}	b_{51}	b_{61}	1.0000	.4000	.5000	.6000	.2000	2.7000
	$a_{22.1}$	$a_{32.1}$	$a_{42.1}$	$a_{52.1}$	$a_{62.1}$		.8400	.1000	.1600	.3200	1.4200
	—	$a_{33.1}$	$a_{43.1}$	$a_{53.1}$	$a_{63.1}$		—	.7500	−.1000	.5000	1.2500
	—	—	$a_{44.1}$	$a_{54.1}$	$a_{64.1}$		—	—	.6400	.6800	1.3800
	1	$b_{32.1}$	$b_{42.1}$	$b_{52.1}$	$b_{62.1}$		1.0000	.1190	.1905	.3810	1.6905
		$a_{33.12}$	$a_{43.12}$	$a_{53.12}$	$a_{63.12}$			.7381	−.1190	.4619	1.0810
		—	$a_{44.12}$	$a_{54.12}$	$a_{64.12}$			—	.6095	.6190	1.1095
		1	$b_{43.12}$	$b_{53.12}$	$b_{63.12}$			1.0000	−.1612	.6258	1.4646
			$a_{44.123}$	$a_{54.123}$	$a_{64.123}$				.5903	.6935	1.2838
			1	$b_{54.123}$	$b_{64.123}$				1.0000	1.1748	2.1748
		1		$b_{53.124}$	$b_{63.124}$			1.0000		.8152	1.8152
	1			$b_{52.134}$	$b_{62.134}$		1.0000			.0602	1.0602
1				$b_{51.234}$	$b_{61.234}$	1.0000				−.9366	.0634

The check column entry is not necessarily equal to the sum of the entries in the row since some of these entries are not recorded, but in such a case it is equal to the sum of the entries as far left as the main diagonal plus those in the column to the top of the matrix. Thus $a_{64.1} = a_{54.1} + a_{44.1} + a_{43.1} + a_{42.1}$.

The method demands the same number of rows as method 2, but there are fewer entries and computations. The back solutions are the same.

Method Four. Abbreviated Method of Single Division

An alternative condensation of method, which can be used whether the equations are symmetric or not, is the abbreviated method of single division which demands less recording than the method of single division. The method consists in computing and recording the entries for the first row and for the first column only of each new matrix. These entries are then used, along with the first rows and columns of previous matrices, to compute entries for the first row and for the first column of the next matrix. For we can write

$$a_{ij.1} = a_{ij} - a_{1j}\, b_{i1}\,,$$

$$a_{ij.12} = a_{ij.1} - a_{2j.1}\, b_{i2.1} = a_{ij} - a_{1j}\, b_{i1} - a_{2j.1}\, b_{i2.1}\,, \text{ etc.}$$

It is necessary to compute these values only as needed. A little practice will enable one to use this method easily. Thus, in Table 4.

$$a_{44.123} = 1.000 - (.6000)(.6000) - (.1600)(.1905) - (-.1190)(-.1612) = .5903\,.$$

TABLE 4. ABBREVIATED METHOD OF SINGLE DIVISION

General						Illustration					
x_1	x_2	x_3	x_4		check	x_1	x_2	x_3	x_4		check
a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	1.0000	.4000	.5000	.6000	.2000	2.7000
a_{12}	a_{22}	a_{32}	a_{42}	a_{52}	a_{62}	.4000	1.0000	.3000	.4000	.4000	2.5000
a_{13}	a_{23}	a_{33}	a_{43}	a_{53}	a_{63}	.5000	.3000	1.0000	.2000	.6000	2.6000
a_{14}	a_{24}	a_{34}	a_{44}	a_{54}	a_{64}	.6000	.4000	.2000	1.0000	.8000	3.0000
1	b_{21}	b_{31}	b_{41}	b_{51}	b_{61}	1.0000	.4000	.5000	.6000	.2000	2.7000
	$a_{22.1}$	$a_{32.1}$	$a_{42.1}$	$a_{52.1}$	$b_{62.1}$		.8400	.1000	.1600	.3200	1.4200
	$a_{23.1}$	—	—	—	—		.1000	—	—	—	—
	$a_{24.1}$	—	—	—	—		.1600	—	—	—	—
	1	$b_{32.1}$	$b_{42.1}$	$b_{52.1}$	$b_{62.1}$		1.0000	.1190	.1905	.3810	1.6905
		$a_{33.12}$	$a_{43.12}$	$a_{53.12}$	$a_{63.12}$			.7381	-.1190	.4619	1.0810
		$a_{34.12}$	—	—	—			-.1190	—	—	—
		1	$b_{43.12}$	$b_{53.12}$	$b_{63.12}$			1.0000	-.1612	.6258	1.4646
			$a_{44.123}$	$a_{54.123}$	$a_{64.123}$				.5903	.6935	1.2838
			1	$b_{54.123}$	$b_{64.123}$				1.0000	1.1748	2.1748
		1		$b_{53.124}$	$b_{63.124}$			1.0000		.8152	1.8152
	1			$b_{52.134}$	$b_{62.134}$		1.0000			.0602	1.0602
1				$b_{51.234}$	$b_{61.234}$	1.0000				-.9366	.0634

Method Five. Abbreviated Method of Single Division — Symmetric

If the matrix of the coefficients is symmetric we may combine the advantages of the method of single division — symmetric and those of the abbreviated method of single division into a new method which we may call the abbreviated method of single division — symmetric. If we combine the omissions of the method of single division — symmetric (the entries below the main diagonal) and those of the abbreviated method of single division (the entries to the right of the first column and below the first row), we find that we have omissions everywhere except in the first row of each matrix (and of course the last row of each matrix which is obtained by dividing the entries of the first row of the matrix by the leading element). In this way we need record but two rows in each matrix and the number of rows and computations, if n is large, is cut down enormously.

It is necessary, however, to adjust our technique for determining the products, as these products must be obtained from the entries of the twin rows of each matrix. The method of using the diagonal to locate the second entry can not now be used after the first matrix, since most of the rows of the matrix do not now appear. However,

$$a_{ij.1} = a_{ij} - a_{1j}\, b_{i1} = a_{ij} - a_{j1}\, b_{i1}\,,$$

TABLE 5. ABBREVIATED METHOD OF SINGLE DIVISION— SYMMETRIC

General						Illustration					
x_1	x_2	x_3	x_4		check	x_1	x_2	x_3	x_4		check
a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	1.0000	.4000	.5000	.6000	.2000	2.7000
—	a_{22}	a_{32}	a_{42}	a_{52}	a_{62}	—	1.0000	.3000	.4000	.4000	2.5000
—	—	a_{33}	a_{43}	a_{53}	a_{63}	—	—	1.0000	.2000	.6000	2.6000
—	—	—	a_{44}	a_{54}	a_{64}	—	—	—	1.0000	.8000	3.0000
1	b_{21}	b_{31}	b_{41}	b_{51}	b_{61}	1.0000	.4000	.5000	.6000	.2000	2.7000
	$a_{22.1}$	$a_{32.1}$	$a_{42.1}$	$a_{52.1}$	$a_{62.1}$		.8400	.1000	.1600	.3200	1.4200
	1	$b_{32.1}$	$b_{42.1}$	$b_{52.1}$	$b_{62.1}$		1.0000	.1190	.1905	.3810	1.6905
		$a_{33.12}$	$a_{43.12}$	$a_{53.12}$	$a_{63.12}$			.7381	−.1190	.4619	1.0810
		1	$b_{43.12}$	$b_{53.12}$	$b_{63.12}$			1.0000	−.1612	.6258	1.4646
			$a_{44.123}$	$a_{54.123}$	$a_{64.123}$				.5903	.6935	1.2838
			1	$b_{54.123}$	$b_{64.123}$				1.0000	1.1748	2.1748
		1		$b_{53.124}$	$b_{63.124}$			1.0000		.8152	1.8152
	1			$b_{52.134}$	$b_{62.134}$		1.0000			.0602	1.0602
1				$b_{51.234}$	$b_{61.234}$	1.0000				−.9366	.0634

$$a_{ij.12} = a_{ij} - a_{j1}\, b_{i1} - a_{2j.1}\, b_{i2.1}$$
$$= a_{ij} - a_{j1}\, b_{i1} - a_{j2.1}\, b_{i2.1}\,,$$
$$a_{ij.123} = a_{ij} - a_{j1}\, b_{i1} - a_{j2.1}\, b_{i2.1} - a_{j3.12}\, b_{i3.12}\,,\ \text{etc.},$$

and the a's in the products are found by taking the entry in the column whose number is the second subscript (j), while the corresponding b's are found by taking the entry in the column whose number is the first subscript (i). Thus the "a" entries used to compute

$$a_{54.123} = a_{54} - a_{41}\, b_{51} - a_{42.1}\, b_{52.1} - a_{43.12}\, b_{43.12}$$

are obtained from the fourth column while the corresponding "b" entries are obtained from the fifth column. Entries in these columns can be easily outlined with the use of a ruler, straight-edge, or even a pencil.

TABLE 6. ABBREVIATED DOOLITTLE SOLUTION

General						Illustration					
x_1	x_2	x_3	x_4		check	x_1	x_2	x_3	x_4		check
a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	1.0000	.4000	.5000	.6000	.2000	2.7000
—	a_{22}	a_{32}	a_{42}	a_{52}	a_{62}	—	1.0000	.3000	.4000	.4000	2.5000
—	—	a_{33}	a_{43}	a_{53}	a_{63}	—	—	1.0000	.2000	.6000	2.6000
—	—	—	a_{44}	a_{54}	a_{64}	—	—	—	1.0000	.8000	3.0000
a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	1.0000	.4000	.5000	.6000	.2000	2.7000
1	b_{21}	b_{31}	b_{41}	b_{51}	b_{61}	1.0000	.4000	.5000	.6000	.2000	2.7000
	$a_{22.1}$	$a_{32.1}$	$a_{42.1}$	$a_{52.1}$	$a_{62.1}$		.8400	.1000	.1600	.3200	1.4200
	1	$b_{32.1}$	$b_{42.1}$	$b_{52.1}$	$b_{62.1}$		1.0000	.1190	.1905	.3810	1.6905
		$a_{33.12}$	$a_{43.12}$	$a_{53.12}$	$a_{63.12}$			.7381	−.1190	.4619	1.0810
		1	$b_{43.12}$	$b_{53.12}$	$b_{63.12}$			1.0000	−.1612	.6258	1.4646
			$a_{44.123}$	$a_{54.123}$	$a_{64.123}$				.5903	.6935	1.2838
			1	$b_{54.123}$	$b_{64.123}$				1.0000	1.1748	2.1748
		1		$b_{53.124}$	$b_{63.124}$			1.0000		.8152	1.8152
	1			$b_{52.134}$	$b_{62.134}$		1.0000			.0602	1.0602
1				$b_{51.234}$	$b_{61.234}$	1.0000				−.9366	.0634

If we lay a ruler to the right of column four and one to the left of column three

$$a_{43.12} = (.2000) - (.5000)\,(.6000) - (.1000)\,(.1905) = -.1190\,.$$

The corresponding check column entry is

$$a_{63.12} = 2.6000 - (.5000)\,(2.7000) - (.1000)\,(1.6905) = 1.0810$$
$$= .7381 - .1190 + .4619\,.$$

The method demands but $4n - 2$ rows, is very compact, and is easily carried out when thoroughly understood. The back solution is similar to that of earlier methods.

Method Six. Abbreviated Doolittle Method

The reader who is familiar with the Doolittle method **(10)** will recognize considerable similarity with the last method. As a matter of fact, the last method is essentially an abbreviated form of the Doolittle method. However, before presenting the Abbreviated Doolittle method, we make an adjustment in the ritual dealing with the first matrix. It would be somewhat simpler, though it would take an additional row, if the first row of the first matrix were repeated at the end of the first n equations just before the first row of b's is inserted. This is done in Table 6 where the Abbreviated Doolittle method is applied to the solution of equations (1) and (2).

The general remarks on method five apply here though the number of rows needed is now $4n - 1$ for the forward and back solutions.

It is customary in the conventional Doolittle solution to make the b's the negatives of the b's of the foregoing solution, but in the method outlined here the negative signs are absorbed in the technique.

The development above shows how the Abbreviated Doolittle method can be obtained as a special case of the method of single division when the matrix of the coefficients is symmetric.

The Abbreviated Doolittle method was outlined essentially by Waugh as early as 1935 **(11)**, though he applied it to the solution of those equations involving correlations. However the idea is immediately applicable to the solution of symmetric equations in general, and, inasmuch as the method does not appear to have attained the recognition which it deserves, it is imperative that its advantages be presented in some detail.

It is possible, of course, to record every step and so to extend the Abbreviated Doolittle method to the conventional Doolittle method, but that is hardly to be recommended as the Doolittle method takes more space, more recording, and is not so accurate. The Abbreviated Doolittle method demands but $4n - 1$ rows, while the conventional Doolittle method demands $2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2} - 1$ additional rows or a total of $\frac{n^2 + 9n - 4}{2}$ rows. For example, Kurtz **(12)** needed 43 rows to present the six equations, the forward solution, and the back solution of his six-variable problem, while with the Abbreviated Doolittle method, this information could be presented in 23

rows. It is obvious, too, that the abbreviated method is desirable in that it saves time used in recording, it eliminates errors which result from excessive recording, and it is theoretically more accurate in that it uses the approximation resulting from the combination of a number of operations rather than the combination of the approximations resulting from the operations. It should be emphasized, too, that the elimination of the unnecessary entries in the forward solution makes it easier to select the entries needed for the computation of the entries in the later matrices. As a matter of fact, one could hardly expect to arrive at a simpler technique than that outlined in the Abbreviated Doolittle method.

The Abbreviated Doolittle method as presented here differs from the conventional Doolittle method not only in the elimination of numerous entries but also in the fact that the division is made by the leading element rather than by its negative. It is the opinion of the author that the division by the leading element is desirable in that (1) it makes possible the carrying on of the back solution in the same form as the computation of the forward solution with a minimum of effort and recording and (2) it preserves the relationship to the method of single division of which the Abbreviated Doolittle method is a special case.

It is advantageous, from a theoretical point of view, to know that the Abbreviated Doolittle method is a special case of the method of single division since the validity of the Abbreviated Doolittle method (and of the conventional Doolittle method itself, since this is essentially the Abbreviated Doolittle method with detailed steps recorded) follows at once from the validity of the method of single division. This is a matter of some importance when it is noted that the writers on the Doolittle method, from Doolittle himself **(10)** down to the present, usually content themselves with a description of the details of the solution rather than a justification of the method. A "proof" can be outlined with a happy choice of notation, but it now appears that validity of the Abbreviated Doolittle method can be demonstrated with little effort since it is a special case of the method of single division. The additional superfluous terms used in the conventional Doolittle method complicate the proof as well as the calculation.

The Doolittle method has been widely recommended as *the* means of solving normal equations when n is greater than three and the matrix is symmetric **(10)**, **(12)**, **(13)**, **(14)**. It now appears, on theoretical as well as practical grounds, that the Abbreviated Doolittle method is to be preferred, — at least if a computing machine capable of performing $a - bc$ operations efficiently is available.

Method Seven. Method of Multiplication and Subtraction

A method which does not involve division at all in the forward solution has lately come into prominence. For example, Rider has used it in his recent text (15, p. 36) and Wren has recently described it (4). If we multiply the first equation of (1) by $-a_{12}$ and the second by a_{11} and add, we get

$$(a_{22}\,a_{11} - a_{12}\,a_{21})\,x_2 + (a_{32}\,a_{11} - a_{12}\,a_{31})\,x_3 + (a_{42}\,a_{11} - a_{12}\,a_{41})\,x_4 = a_{52}\,a_{11} - a_{12}\,a_{51}\,,$$

and if we let $A_{ij.1} = a_{ij}\,a_{11} - a_{1j}\,a_{i1}$, this appears as

$$A_{22.1}\,x_2 + A_{32.1}\,x_3 + A_{42.1}\,x_4 = A_{52.1}\,.$$

TABLE 7. METHOD OF MULTIPLICATION AND SUBTRACTION

General						Illustration					
x_1	x_2	x_3	x_4		check	x_1	x_2	x_3	x_4		check
a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	1.0000	.4000	.5000	.6000	.2000	2.7000
a_{12}	a_{22}	a_{32}	a_{42}	a_{52}	a_{62}	.4000	1.0000	.3000	.4000	.4000	2.5000
a_{13}	a_{23}	a_{33}	a_{43}	a_{53}	a_{63}	.5000	.3000	1.0000	.2000	.6000	2.6000
a_{14}	a_{24}	a_{34}	a_{44}	a_{54}	a_{64}	.6000	.4000	.2000	1.0000	.8000	3.0000
	$A_{22.1}$	$A_{32.1}$	$A_{42.1}$	$A_{52.1}$	$A_{62.1}$		.8400	.1000	.1600	.3200	1.4200
	$A_{23.1}$	$A_{33.1}$	$A_{43.1}$	$A_{53.1}$	$A_{63.1}$		.1000	.7500	−.1000	.5000	1.2500
	$A_{24.1}$	$A_{34.1}$	$A_{44.1}$	$A_{54.1}$	$A_{64.1}$		.1600	−.1000	.6400	.6800	1.3800
		$A_{33.12}$	$A_{43.12}$	$A_{53.12}$	$A_{63.12}$			.6200	−.1000	.3880	.9080
		$A_{34.12}$	$A_{44.12}$	$A_{54.12}$	$A_{64.12}$			−.1000	.5120	.5200	.9320
			$A_{44.123}$	$A_{54.123}$	$A_{64.123}$				.30744	.36120	.66864
			1	$B_{54.123}$	$B_{64.123}$				1.0000	1.1749	2.1749
		1		$B_{53.124}$	$B_{63.124}$			1.0000		.8153	1.8153
	1			$B_{52.134}$	$B_{62.134}$		1.0000			.0601	1.0601
1				$B_{51.234}$	$B_{61.234}$	1.0000				−.9366	.0634

A similar elimination involving the first and third equations gives

$$A_{23.1}\,x_2 + A_{33.1}\,x_3 + A_{43.1}\,x_4 = A_{53.1}\,,$$

and similarly

$$A_{24.1}\,x_2 + A_{34.1}\,x_3 + A_{44.1}\,x_4 = A_{54.1}\,.$$

The four equations in four unknowns have thus been reduced, by multiplication and subtraction, to three equations in three unknowns. The process can be repeated with the resulting

$$A_{33.12}\,x_3 + A_{43.12}\,x_4 = A_{53.12}\,,$$

$$A_{34.12}\,x_3 + A_{44.12}\,x_4 = A_{54.12}\,,$$

with

$$A_{ij.12} = A_{ij.1} A_{22.1} - A_{i2.1} A_{2j.1} .$$

Again

$$A_{44.123} x_4 = A_{54.123} ,$$

with

$$A_{ij.123} = A_{ij.12} A_{33.12} - A_{i3.12} A_{3j.12} .$$

So

$$x_4 = \frac{A_{54.123}}{A_{44.123}} = B_{54.123} ,$$

$$x_3 = \frac{1}{A_{33.12}} [A_{53.12} - A_{43.12} B_{54.123}] , \text{ etc.}$$

The forward solution is simpler and the back solution slightly more involved than methods previously presented. The back solution involves a division for each entry though there are no divisions in the forward solution. The solution demands $\frac{n(n+3)}{2}$ rows.

The chief advantage of the method is the $ab - cd$ technique which is used in getting the items in each successive matrix. Starting with a given term in a matrix, multiply by the upper left entry of the matrix and subtract the product of the entries at the top of the matrix column and the one at the left of the matrix row. The illustration is given in Table 7.

Method Eight. Method of Multiplication and Subtraction—Symmetric

If the matrix of the coefficients is symmetric, it is possible to leave the entries below the diagonal as blanks and to proceed somewhat as in method three (15, pp. 37-38). Any item in a matrix is now multiplied by the leading item in the matrix, and from this product is subtracted the product of the item heading the column with the item heading the column which is obtained by going to the left until the main diagonal is reached. The details of the numerical solution for equations (1) and (2) are given in Table 8.

Thus

$$A_{43.1} = (.2000)(1.0000) - (.5000)(.6000) = -.1000$$

and

$$A_{43.12} = (-.1000)(.8400) - (.1000)(.1600) = -.1000 .$$

Method Nine. Abbreviated Method of Multiplication and Subtraction

It is not necessary to record the entries except those in the first row and in the first column of each matrix. Each value $A_{ij.\cdots}$ is then

TABLE 8. METHOD OF MULTIPLICATION AND SUBTRACTION—SYMMETRIC

General						Illustration					
x_1	x_2	x_3	x_4		check	x_1	x_2	x_3	x_4		check
a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	1.0000	.4000	.5000	.6000	.2000	2.7000
—	a_{22}	a_{32}	a_{42}	a_{52}	a_{62}	—	1.0000	.3000	.4000	.4000	2.5000
—	—	a_{33}	a_{43}	a_{53}	a_{63}	—	—	1.0000	.2000	.6000	2.6000
—	—	—	a_{44}	a_{54}	a_{64}	—	—	—	1.0000	.8000	3.0000
	$A_{22.1}$	$A_{32.1}$	$A_{42.1}$	$A_{52.1}$	$A_{62.1}$		.8400	.1000	.1600	.3200	1.4200
	—	$A_{33.1}$	$A_{43.1}$	$A_{53.1}$	$A_{63.1}$		—	.7500	−.1000	.5000	1.2500
	—	—	$A_{44.1}$	$A_{54.1}$	$A_{64.1}$		—	—	.6400	.6800	1.3800
		$A_{33.12}$	$A_{43.12}$	$A_{53.12}$	$A_{63.12}$			.6200	−.1000	.3880.	9080
		—	$A_{44.12}$	$A_{54.12}$	$A_{64.12}$			—	.5120	.5200	.9320
			$A_{44.123}$	$A_{54.123}$	$A_{64.123}$				.30744	.36120	.66864
			1	$B_{54.123}$	$B_{64.123}$				1.0000	1.1749	2.1749
		1		$B_{53.124}$	$B_{63.124}$			1.0000		.8153	1.8153
	1			$B_{52.134}$	$B_{62.134}$		1.0000			.0601	1.0601
1				$B_{51.234}$	$B_{61.234}$	1.0000				−.9366	.0634

TABLE 9. ABBREVIATED METHOD OF MULTIPLICATION AND SUBTRACTION

General						Illustration					
x_1	x_2	x_3	x_4		check	x_1	x_2	x_3	x_4		check
a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	1.0000	.4000	.5000	.6000	.2000	2.7000
a_{12}	a_{22}	a_{32}	a_{42}	a_{52}	a_{62}	4.000	1.0000	.3000	.4000	.4000	2.5000
a_{13}	a_{23}	a_{33}	a_{43}	a_{53}	a_{63}	.5000	.3000	1.0000	.2000	.6000	2.6000
a_{14}	a_{24}	a_{34}	a_{44}	a_{54}	a_{64}	.6000	.4000	.2000	1.0000	.8000	3.0000
	$A_{22.1}$	$A_{32.1}$	$A_{42.1}$	$A_{52.1}$	$A_{62.1}$		.8400	.1000	.1600	.3200	1.4200
	$A_{23.1}$	—	—	—	—		.1000	—	—	—	—
	$A_{24.1}$	—	—	—	—		.1600	—	—	—	—
		$A_{33.12}$	$A_{43.12}$	$A_{53.12}$	$A_{63.12}$			.6200	−.1000	.3880	.9080
		$A_{34.12}$	—	—	—			−.1000	—	—	—
			$A_{44.123}$	$A_{54.123}$	$A_{64.123}$				.30744	.36120	.66864
			1	$B_{54.123}$	$B_{64.123}$				1.0000	1.1749	2.1749
		1		$B_{53.124}$	$B_{63.124}$			1.0000		.8153	1.8153
	1			$B_{52.134}$	$B_{62.134}$		1.0000			.0601	1.0601
1				$B_{51.234}$	$B_{61.234}$	1.0000				−.9366	.0634

obtained by multiplying by the leading term and subtracting the appropriate product in each matrix. Thus

$$A_{ij.1} = a_{ij}\, a_{11} - a_{i1}\, a_{1j}\,,$$

$$A_{ij.12} = (a_{ij}\, a_{11} - a_{i1}\, a_{1j})\, A_{22.1} - A_{2j.1}\, A_{i2.1}\,,$$

and

$$A_{ij.123} = [\,(a_{ij}\, a_{11} - a_{i1}\, a_{1j})\, A_{22.1} - A_{2j.1}\, A_{i2.1}]\, A_{33.12} - A_{3j.12}\, A_{i3.12}\,.$$

The process of computation is easily carried out with a modern computing machine. The details of the solution are given in Table 9. Thus

$$A_{43.12} = [\,(.2000)\,(1.0000) - (.5000)\,(.6000)\,]\,.8400 - (.1000)\,(.1600) = -.1000$$

and

$$A_{44.123} = \{[\,(1.0000)\,(1.0000) - (.6000)\,(.6000)\,]\,.8400 - (.1600)\,(.1600)\,\}\,(.6200) - (.1000)\,(.1000) \quad .30744\,.$$

The method demands the same number of rows as method seven but fewer entries are used.

Method Ten. Abbreviated Method of Multiplication and Subtraction — Symmetric

In case the matrix is symmetric it is only necessary to record the first row of each matrix since the columnar entries duplicate the row entries. The values of the first rows of each matrix are determined from the matrices above. The multipliers are obtained by taking the products of the terms in the *i*'th and *j*'th columns. These columns may be outlined with the use of a straight-edge, though care must be taken not to cover up the first entry in the row, for this is needed for multiplication. For example, if $A_{43.\cdots}$ is desired, then the entries, with the exception of the leading items, are all in column 3 and column 4. The solution is given in Table 10 with the omission of the first row in the second matrix, the insertion of which is discussed later.

Thus

$$A_{43.12} = [\,(.2000)\,(1.0000) - (.5000)\,(.6000)\,]\,(.8400) - (.1000)\,(.1600) = -.1000$$

and

$$A_{44.123} = \{[\,(1.0000)\,(1.0000) - (.6000)^2\,]\,(.8400) - (.1600)^2\}\times (.6200) - (-.1000)^2 = .30744\,.$$

TABLE 10. ABBREVIATED METHOD OF MULTIPLICATION AND SUBTRACTION—SYMMETRIC

General						Illustration					
x_1	x_2	x_3	x_4		check	x_1	x_2	x_3	x_4		check
a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	1.0000	.4000	.5000	.6000	.2000	2.7000
—	a_{22}	a_{32}	a_{42}	a_{52}	a_{62}	—	1.0000	.3000	.4000	.4000	2.5000
—	—	a_{33}	a_{43}	a_{53}	a_{63}	—	—	1.0000	.2000	.6000	2.6000
—	—	—	a_{44}	a_{54}	a_{64}	—	—	—	1.0000	.8000	3.0000
a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	1.0000	.4000	.5000	.6000	.2000	2.7000
	$A_{22.1}$	$A_{32.1}$	$A_{42.1}$	$A_{52.1}$	$A_{62.1}$		.8400	.1000	.1600	.3200	1.4200
		$A_{33.12}$	$A_{43.12}$	$A_{53.12}$	$A_{63.12}$			.6200	−.1000	.3880	.9080
			$A_{44.123}$	$A_{54.123}$	$A_{64.123}$				.30744	.36120	.66864
			1	$B_{54.123}$	$B_{64.123}$				1.0000	1.1749	2.1749
		1		$B_{53.124}$	$B_{63.124}$			1.0000		.8153	1.8153
	1			$B_{52.134}$	$B_{62.134}$		1.0000			.0601	1.0601
1				$B_{51.234}$	$B_{61.234}$	1.0000				−.9366	.0634

This method has not, to my knowledge, been presented before. It follows logically from the method of multiplication and subtraction just as the Abbreviated Doolittle method follows from the method of single division. This method is very compact and demands but $3n - 1$ rows of which n are demanded for the statement of the problem and n for the statement of the solution. The technique of the method is easily learned. The forward solution is more compact than the Abbreviated Doolittle method while the back solution is somewhat longer owing to the fact that a final division must be made. However, there are no divisions (though there are multiplications) in the forward solution, and the actual solution, aside from the n rows needed to express the problem and the n rows stating the result, takes about half as much space as the Abbreviated Doolittle solution.

For those who use the method only occasionally it may be better to repeat the first row at the beginning of the second matrix (see Table 10) as we did in the Abbreviated Doolittle solution. The ritual can then be more easily remembered for it is necessary only to locate columns i and j, to multiply a_{ij} by the leading element of the first row of the second part of the solution, to subtract the product of the entries in the i'th and j'th columns in the row, to multiply the result by the leading element in the next row, to subtract the product of the elements of the i'th and j'th columns in this row, etc. The space devoted to the solution is then divided into three equal parts of n rows, (1) the statement of the problem, (2) the details of the forward solution, and (3) the statement of the results. For brevity we shall refer to this method as the "compact" method.

THE SOLUTION OF RELATED EQUATIONS

There are some additional points which should be made in connection with the solution of equations by the methods herein described. It should be noted first that the forward solution of such equations as

$$\begin{aligned} a_{11}x_1 + a_{21}x_2 + a_{31}x_3 &= a_{51}, \\ a_{12}x_1 + a_{22}x_2 + a_{32}x_3 &= a_{52}, \\ a_{13}x_1 + a_{23}x_2 + a_{33}x_3 &= a_{53}, \end{aligned} \tag{6}$$

and

$$\begin{aligned} a_{11}x_1 + a_{21}x_2 &= a_{51}, \\ a_{12}x_1 + a_{22}x_2 &= a_{52}, \end{aligned} \tag{7}$$

are presented in each of the methods of solving (1). It is necessary only to cover up the fourth column and complete the back solution. For example, from the forward portion of the Abbreviated Doolittle method (Table 6), we have:

$$\begin{aligned} x_3 &= b_{53.12}, & x_3 &= .6258, \\ x_2 &= b_{52.1} - b_{32.1}\,b_{53.12}\ (= b_{52.13}), & x_2 &= .3065, \\ x_1 &= b_{51} - b_{31}\,b_{53.12} - b_{21}\,b_{52.13}, & x_1 &= -.2355. \end{aligned}$$

Similarly, from the forward solution of Table 10, we have

$$\begin{aligned} x_3 &= \frac{A_{53.12}}{A_{33.12}} = B_{53.12}, & x_3 &= .6258, \\ x_2 &= \frac{1}{A_{22.1}}[A_{52.1} - A_{32.1}\,B_{53.12}]\ (= B_{52.13}), & x_2 &= .3065, \\ x_1 &= \frac{1}{a_{11}}[a_{51} - a_{31}\,B_{53.12} - a_{21}\,B_{52.13}]. & x_1 &= -.2355. \end{aligned}$$

The solution is verified by substituting in the equations. Thus

$$\begin{aligned} x_1 + .4x_2 + .5x_3 &= .20000 \\ .4x_1 + x_2 + .3x_3 &= .40004 \\ .5x_1 + .3x_2 + x_3 &= .60000 \end{aligned} \quad \text{when} \quad \left\{ \begin{aligned} x_1 &= -.2355 \\ x_2 &= .3065 \\ x_3 &= .6258 \end{aligned} \right\}.$$

The reader is referred to Kurtz (12) for a more general discussion dealing with the solution of related equations.

THE SOLUTION OF GROUPS OF EQUATIONS

It is sometimes desired to solve groups of equations in which the variables have the same coefficients, but the constants are different.

TABLE 11. GROUPS OF EQUATIONS — ABBREVIATED DOOLITTLE

x_1	x_2	x_3	x_4				check	x_1	x_2	x_3	x_4				check
a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	a_{71}	a_{81}	1.0000	.4000	.5000	.6000	.2000	.4000	.8000	3.9000
—	a_{22}	a_{32}	a_{42}	a_{52}	a_{62}	a_{72}	a_{82}	—	1.0000	.3000	.4000	.4000	.5000	.6000	3.6000
—	—	a_{33}	a_{43}	a_{53}	a_{63}	a_{73}	a_{83}	—	—	1.0000	.2000	.6000	.6000	.4000	3.6000
—	—	—	a_{44}	a_{54}	a_{64}	a_{74}	a_{84}	—	—	—	1.0000	.8000	.7000	.2000	3.9000
a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	a_{71}	a_{81}	1.0000	.4000	.5000	.6000	.2000	.4000	.8000	3.9000
1	b_{21}	b_{31}	b_{41}	b_{51}	b_{61}	b_{71}	b_{81}	1.0000	.4000	.5000	.6000	.2000	.4000	.8000	3.9000
	$a_{22.1}$	$a_{32.1}$	$a_{42.1}$	$a_{52.1}$	$a_{62.1}$	$a_{72.1}$	$a_{82.1}$		.8400	.1000	.1600	.3200	.3400	.2800	2.0400
	1	$b_{32.1}$	$b_{42.1}$	$b_{52.1}$	$b_{62.1}$	$b_{72.1}$	$b_{82.1}$		1.0000	.1190	.1905	.3810	.4048	.3333	2.4286
		$a_{33.12}$	$a_{43.12}$	$a_{53.12}$	$a_{63.12}$	$a_{73.12}$	$a_{83.12}$			.7381	−.1190	.4619	.3595	−.0333	1.4011
		1	$b_{43.12}$	$b_{53.12}$	$b_{63.12}$	$b_{73.12}$	$b_{83.12}$			1.0000	−.1612	.6258	.4871	−.0451	1.9064
			$a_{44.123}$	$a_{54.123}$	$a_{64.123}$	$a_{74.123}$	$a_{84.123}$				.5903	.6935	.4532	−.3387	1.3983
			1	$b_{54.123}$	$b_{64.123}$	$b_{74.123}$	$b_{84.123}$				1.0000	1.1748	.7677	−.5738	2.3688
		1		$b_{53.124}$	$b_{63.124}$	$b_{73.124}$	$b_{83.124}$			1.0000		.8152	.6109	−.1376	2.2883
	1			$b_{52.134}$	$b_{62.134}$	$b_{72.134}$	$b_{82.134}$		1.0000			.0602	.1859	.4590	1.7050
1				$b_{51.234}$	$b_{61.234}$	$b_{71.234}$	$b_{81.234}$	1.0000				−.9366	−.4404	1.0295	.6526

TABLE 12. GROUPS OF EQUATIONS—COMPACT METHOD

x_1	x_2	x_3	x_4				check	x_1	x_2	x_3	x_4				check
a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	a_{71}	a_{81}	1.0000	.4000	.5000	.6000	.2000	.4000	.8000	3.9000
—	a_{22}	a_{32}	a_{42}	a_{52}	a_{62}	a_{72}	a_{82}	—	1.0000	.3000	.4000	.4000	.5000	.6000	3.6000
—	—	a_{33}	a_{43}	a_{53}	a_{63}	a_{73}	a_{83}	—	—	1.0000	.2000	.6000	.6000	.4000	3.6000
—	—	—	a_{44}	a_{54}	a_{64}	a_{74}	a_{84}	—	—	—	1.0000	.8000	.7000	.2000	3.9000
a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	a_{71}	a_{81}	1.0000	.4000	.5000	.6000	.2000	.4000	.8000	3.9000
	$A_{22.1}$	$A_{32.1}$	$A_{42.1}$	$A_{52.1}$	$A_{62.1}$	$A_{72.1}$	$A_{82.1}$		.8400	.1000	.1600	.3200	.3400	.2800	2.0400
		$A_{33.12}$	$A_{43.12}$	$A_{53.12}$	$A_{63.12}$	$A_{73.12}$	$A_{83.12}$			.6200	−.1000	.3880	.3020	−.0280	1.1820
			$A_{44.123}$	$A_{54.123}$	$A_{64.123}$	$A_{74.123}$	$A_{84.123}$				.30744	.36120	.23604	−.17640	.72828
			1	$B_{54.123}$	$B_{64.123}$	$B_{74.123}$	$B_{84.123}$				1.0000	1.1749	.7678	−.5738	2.3689
		1		$B_{53.124}$	$B_{63.124}$	$B_{73.124}$	$B_{83.124}$			1.0000		.8153	.6109	−.1377	2.2885
	1			$B_{52.134}$	$B_{62.134}$	$B_{72.134}$	$B_{82.134}$		1.0000			.0601	.1858	.4590	1.7049
1				$B_{51.234}$	$B_{61.234}$	$B_{71.234}$	$B_{81.234}$	1.0000				−.9366	−.4404	1.0295	.6525

Thus the values of a_{51}, a_{52}, a_{53}, a_{54}, in (2) might be: (*a*) .2, .4, .6, .8; (*b*) .4, .5, .6, .7, and (*c*) .8, .6, .4, .2.

It is possible, in any of the solutions indicated above, to insert additional columns and to carry along the solutions simultaneously. The check column should show the sum of the entries in the row. This method is advised in case a few groups of equations are involved. The solution which uses the Abbreviated Doolittle method is given in Table 11.

The results are obtained by the more accurate "Compact" method in Table 12.

The Solution of Groups of Equations — Fisher Method

In case the number of groups is large, however, it is better to use a more general method. If each group contains n equations in n unknowns, a special column can be reserved for the coefficients of $a_{n+1,1}$, one for the coefficients of $a_{n+2,1}$, etc. These coefficients are either 1 or 0 and constitute a unit diagonal matrix. Thus equations (1) can be written symbolically as

x_1	x_2	x_3	x_4	a_{51}	a_{52}	a_{53}	a_{54}
a_{11}	a_{21}	a_{31}	a_{41}	1	0	0	0
a_{12}	a_{22}	a_{32}	a_{42}	0	1	0	0
a_{13}	a_{23}	a_{33}	a_{43}	0	0	1	0
a_{14}	a_{24}	a_{34}	a_{44}	0	0	0	1

(8)

and the solution can be worked out in terms of a_{51}, a_{52}, a_{53}, a_{54}, by the methods just described. It is necessary only to insert any desired values of a_{5j} to obtain the solution.

It appears that this method was first outlined by R. A. Fisher (16, p. 150). It can be combined with any of the methods described above. The author has shown how it can be combined with the conventional Doolittle solution (17). In Tables 13 and 14 are presented the Abbreviated Doolittle solution and the "Compact" solution of equations (2) for any a_{5j}. The check column, a_{6j} is as usual the sum of the entries in the completed row.

TABLE 13. ABBREVIATED DOOLITTLE-FISHER METHOD

x_1	x_2	x_3	x_4	a_{51}	a_{52}	a_{53}	a_{54}	check a_{6j}
1.0000	.4000	.5000	.6000	1	0	0	0	3.5000
—	1.0000	.3000	.4000	0	1	0	0	3.1000
—	—	1.0000	.2000	0	0	1	0	3.0000
—	—	—	1.0000	0	0	0	1	3.2000
1.0000	.4000	.5000	.6000	1.0000	0	0	0	3.5000
1.0000	.4000	.5000	.6000	1.0000	0	0	0	3.5000
	.8400	.1000	.1600	−.4000	1.0000	0	0	1.7000
	1.0000	.1190	.1905	−.4762	1.1905	0	0	2.0238
		.7381	−.1190	−.4524	−.1190	1.0000	0	1.0476
		1.0000	−.1612	−.6129	−.1612	1.3548	0	1.4193
			.5903	−.5966	−.2097	.1612	1.0000	.9451
			1.0000	−1.0107	−.3552	.2731	1.6941	1.6011
		1.0000		−.7758	−.2185	1.3988	.2731	1.6774
	1.0000			−.1913	1.2842	−.2185	−.3552	1.5192
1.0000				2.0708	−.1913	−.7759	−1.0109	1.0930

TABLE 14. COMPACT-FISHER METHOD

x_1	x_2	x_3	x_4	a_{51}	a_{52}	a_{53}	a_{54}	check a_{6j}
1.0000	.4000	.5000	.6000	1	0	0	0	3.5000
—	1.0000	.3000	.4000	0	1	0	0	3.1000
—	—	1.0000	.2000	0	0	1	0	3.0000
—	—	—	1.0000	0	0	0	1	3.2000
1.0000	.4000	.5000	.6000	1.0000	0	0	0	3.5000
	.8400	.1000	.1600	−.4000	1.0000	0	0	1.7000
		.6200	−.1000	−.3800	−.1000	.8400	0	.8800
			.30744	−.3108	−.1092	.0840	.5208	.49224
			1.0000	−1.0109	−.3552	.2732	1.6940	1.6011
		1.0000		−.7759	−.2186	1.3989	.2732	1.6776
	1.0000			−.1913	1.2842	−.2186	−.3552	1.5191
1.0000				2.0710	−.1913	−.7759	−1.0109	1.0929

We have then

$$
\begin{aligned}
x_1 &= 2.0710a_{51} - .1913a_{52} - .7759a_{53} - 1.0109a_{54} \\
x_2 &= -.1913a_{51} + 1.2842a_{52} - .2186a_{53} - .3552a_{54} \\
x_3 &= -.7759a_{51} - .2186a_{52} + 1.3989a_{53} + .2732a_{54} \\
x_4 &= -1.0109a_{51} - .3552a_{52} + .2732a_{53} + 1.6940a_{54}\,.
\end{aligned} \tag{9}
$$

The values of a_{5j} are then inserted to obtain values of x_i. For example, when $a_{51} = .2$, $a_{52} = .4$, $a_{53} = .6$, $a_{54} = .8$, we have

$$x_1 = -.9366, \quad x_2 = .0601, \quad x_3 = .8153, \quad x_4 = 1.1749$$

and these results agree with those of Tables 11 and 12.

The Inverse of a Matrix

The solutions just above give a method for finding the inverse of a matrix since they provide the matrix A^{-1} in

$$AA^{-1} = \mathrm{I}.$$

The method above can be used whether A is a square matrix, or not. In case A is a square symmetric matrix it is possible to shorten the Fisher solution appreciably by omitting the columns on the right. The matrix A^{-1} is symmetric, and this fact can be used to eliminate many of the entries. Furthermore, the intermediate equations take the form, in the "Compact" method,

$$\begin{aligned}
a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 &= y_1 \\
A_{22.1}x_2 + A_{32.1}x_3 + A_{42.1}x_4 &= C_{12}y_1 + a_{11}y_2 \\
A_{33.12}x_3 + A_{43.12}x_4 &= C_{13}y_1 + C_{23}y_2 + a_{11}A_{22.1}y_3 \\
A_{44.123}x_4 &= C_{14}y_1 + C_{24}y_2 + C_{34}y_3 \\
&\quad + a_{11}A_{22.1}A_{33.12}y_4 .
\end{aligned} \tag{10}$$

It is possible to find formal expressions for the C's in terms of the A's, but the method outlined below avoids this. If the equations (10) are solved for x_1, x_2, x_3, x_4, we get

$$\begin{aligned}
x_1 &= D_{11}y_1 + D_{21}y_2 + D_{31}y_3 + D_{41}y_4 \\
x_2 &= D_{12}y_1 + D_{22}y_2 + D_{32}y_3 + D_{42}y_4 \\
x_3 &= D_{13}y_1 + D_{23}y_2 + D_{33}y_3 + D_{43}y_4 \\
x_4 &= D_{14}y_1 + D_{24}y_2 + D_{34}y_3 + D_{44}y_4 ,
\end{aligned} \tag{11}$$

where the D's form the symmetric matrix $D = A^{-1}$ with $D_{ij} = D_{ji}$. Now from the last equation of (10)

$$D_{44} = \frac{a_{11}A_{22.1}A_{33.12}}{A_{44.123}},$$

so that, by solving for D_{4j} in the third, second, and first equations of (10), we get

$$D_{43} = \frac{1}{A_{33.12}} [-A_{43.12} D_{44}]$$

$$D_{42} = \frac{1}{A_{22.1}} [-A_{32.1} D_{43} - A_{42.1} D_{44}]$$

$$D_{41} = \frac{1}{a_{11}} [-a_{21} D_{42} - a_{31} D_{43} - a_{41} D_{44}] .$$

We next fill in the values D_{14}, D_{24}, D_{34}, since $D_{ij} = D_{ji}$ and note

$$D_{33} = \frac{1}{A_{33.12}} [a_{11} A_{22.1} - A_{43.12} D_{34}]$$

$$D_{32} = \frac{1}{A_{22.1}} [-A_{32.1} D_{33} - A_{42.1} D_{34}]$$

$$D_{31} = \frac{1}{a_{11}} [-a_{21} D_{32} - a_{31} D_{33} - A_{41} D_{34}]$$

with $D_{13} = D_{31}$, $D_{12} = D_{21}$.

Similarly,

$$D_{22} = \frac{1}{A_{22.1}} [a_{11} - A_{32.1} D_{23} - A_{24.1} D_{24}]$$

$$D_{21} (= D_{12}) = \frac{1}{a_{11}} [-a_{21} D_{22} - A_{31} D_{23} - a_{41} D_{24}]$$

$$D_{11} = \frac{1}{a_{11}} [1 - a_{21} D_{12} - a_{31} D_{13} - a_{41} D_{14}] .$$

TABLE 15(a). THE INVERSE OF A SQUARE SYMMETRIC MATRIX. COMPACT METHOD

1.0000	.4000	.5000	.6000				
—	1.0000	.3000	.4000				
—	—	1.0000	.2000				
—	—	—	1.0000				
(1.0000) 1.0000	.4000	.5000	.6000	2.0710	–.1913	–.7759	–1.0109
	(1.0000) .8400	.1000	.1600	–.1913	1.2842	–.2186	–.3552
		(.8400) .6200	–.1000	–.7759	–.2186	1.3989	.2732
			(.5208) .30744	–1.0109	–.3552	.2732	1.6940

An illustration is given in Table 15(a) where the compact method is used.

The first four columns are characteristic of the forward solution of the compact method. In addition, in the diagonal of the second matrix, I have inserted in parentheses the values, 1, a_{11}, $a_{11}A_{22.1}$, $a_{11}A_{22.1} \times A_{33.12}$. It follows at once that

$$D_{44} = \frac{.5208}{.30744} = 1.6940; \quad D_{43} = \frac{1}{.6200}[-(-.1000)(1.6940)] = .2732$$

$$D_{33} = \frac{1}{.6200}[.8400 - (-.1000)(.2732)], \text{ etc.}$$

The method is not at all difficult to apply once the formulas above are thoroughly understood. The type of operation is similar to that used in the usual back solution.

An alternative method is provided in Table 15(b) where the first two matrices agree with those of Table 15(a). As before, $D_{44} = \frac{.5208}{.30744} = 1.6940$ and $D_{43} = \frac{1}{.6200}[-(-.1000)(1.6940)] = .2732$.

The value $D_{34} = .2732$ is then inserted and the value of D_{42} found, essentially, by taking products of terms in the same columns. The

TABLE 15(*b*). THE INVERSE OF A SQUARE SYMMETRIC MATRIX. COMPACT METHOD

1.0000	.4000	.5000	.6000
—	1.0000	.3000	.4000
—	—	1.0000	.2000
—	—	—	1.0000
(1.0000) 1.0000	.4000	.5000	.6000
	(1.0000) .8400	.1000	.1600
		(.8400) .6200	−.1000
			(.5208) .30744
2.0710	−.1913	−.7759	−1.0109
−.1913	1.2842	−.2186	−.3552
−.7759	−.2185	1.3989	.2732
−1.0109	−.3552	.2732	1.6940

general idea is to get the higher vertical terms in each column and, as soon as one is available, write it in the proper row so that the next element can be found. When all the elements in a given row are found, move on to the next row, etc. This, too, is a simple technique once it is thoroughly understood.

A similar scheme can be worked out for the Abbreviated Doolittle method. In this case the intermediate equations have the form

$$\begin{aligned} x_1 + b_{21}\,x_2 + b_{31}\,x_3 + b_{41}\,x_4 &= \frac{1}{a_{11}}\,y_1 \\ x_2 + b_{32.1}\,x_3 + b_{42.1}\,x_4 &= E_{12}\,y_1 + \frac{1}{a_{22.1}}\,y_2 \\ x_3 + b_{43.12}\,x_4 &= E_{13}\,y_1 + E_{23}\,y_2 + \frac{1}{a_{33.12}}\,y_3 \\ x_4 &= E_{14}\,y_1 + E_{24}\,y_2 + E_{34}\,y_3 + \frac{1}{a_{44.123}}\,y_4 \end{aligned} \tag{12}$$

so that

$$\begin{aligned} D_{44} &= \frac{1}{a_{44.123}}\,, \quad D_{43} = -b_{43.12}\,D_{44} = D_{34} \\ D_{42} &= -(b_{32.1}\,D_{43} + b_{42.1}\,D_{44}) = D_{24} \\ D_{41} &= -(b_{21}\,D_{42} + b_{31}\,D_{43} + b_{41}\,D_{44}) = D_{14} \\ D_{33} &= \frac{1}{a_{33.12}} - b_{43.12}\,D_{34}\,, \text{ etc.} \end{aligned}$$

The numerical illustration is given in Table 16, where the two alternative solutions are shown. The values $\frac{1}{a_{11}}, \frac{1}{a_{22.1}}, \frac{1}{a_{33.12}}, \frac{1}{a_{44.123}}$ are inserted above a_{11}, $a_{22.1}$, $a_{33.12}$, $a_{44.123}$. The first solution on the right is similar to the solution in Table 15 (a) while the second solution at the bottom is similar in method to the solution of Table 15 (b).

This version of the method for finding the inverse of a square symmetric matrix has been given, essentially, by Waugh (11) who made improvements on the technique previously worked out by Horst (18) (19). Waugh used the Abbreviated Doolittle method and applied it to the correlation matrix and showed how to compute the large number of statistical quantities which can be computed from the inverse of the correlation matrix. Dunlap (20, pp. 119-123) has presented a similar method which he attributes to Cureton.

TABLE 16. THE INVERSE OF A SQUARE SYMMETRIC MATRIX—ABBREVIATED DOOLITTLE METHOD

1.0000	.4000	.5000	.6000				
—	1.0000	.3000	.4000				
—	—	1.0000	.2000				
—	—	—	1.0000				
1.0000	.4000	.5000	.6000				
(1.0000) 1.0000	.4000	.5000	.6000	2.0710	−.1913	−.7759	−1.0109
	.8400	.1000	.1600				
	(1.1905) 1.0000	.1190	.1905	−.1913	1.2842	−.2185	−.3552
		.7381	−.1190				
		(1.3548) 1.0000	−.1612	−.7759	−.2185	1.3988	.2731
			(1.6941) .5903	−1.0109	−.3552	.2731	1.6941
2.0710	−.1913	−.7759	−1.0109				
−.1913	1.2842	−.2185	−.3552				
−.7759	−.2185	1.3988	.2731				
−1.0109	−.3552	.3731	1.6941				

Other methods of finding the inverse of a matrix have been published recently (9) (21). These combine one of the methods outlined at the beginning of the paper with a scheme, which seems to be due to Aitken (9), which carries out the back solution by "forward" means.

Solution of Non-Symmetric Equations by Symmetric Methods

Aitken has shown (9) that it is possible to reduce non-symmetric equations to equivalent symmetric equations. If the equations are represented by $Ax = B$, then by premultiplying by A' we have the equivalent

$$A'Ax = A'B,$$

with $A'A$ a symmetric matrix.

Conclusion

Various known methods of solving equations have been presented in such a fashion as to bring out their relations with other methods. By gradual steps the method of single division leads to an abbreviated

version of the Doolittle method while, by corresponding steps, the method of multiplication and subtraction leads to a new compact method. These methods are applied to such problems as the solution of groups of equations and the evaluation of the inverse of a matrix. The evaluation of determinants, of multiple correlation coefficients, of regression coefficients, of linear forms, etc., will be discussed in later papers.

The bibliography attached, while extensive, does not pretend to be complete. Some of the references themselves provide extensive bibliography and, in this way, it is possible to contact, from the references below, much of the work done in this field.

REFERENCES

1. Salisbury, F. S. A simplified method of computing multiple correlation constants. *J. educ. Psych.*, 1929, **20**, 44-52.
2. Griffin, H. D. On partial correlation versus partial regression for obtaining the multiple regression equations. *J. educ. Psych.*, 1931, **22**, 35-44. (Particularly valuable for the bibliography.)
3. Thompson, G. H. Some points of mathematical technique in the factorial analysis of ability. *J. educ. Psych.*, 1936, **27**, 37-54.
4. Wren, F. L. The calculation of partial and multiple coefficients of regression and correlation. *J. educ. Psych.*, 1938, **29**, 695-700.
5. Huntington, E. V. Curve-fitting by the method of least squares and the method of moments. Handbook of mathematical statistics. Chap. IV. New York: Houghton Mifflin Co.
6. Brolyer, C. R. Another short method for calculating β weights and the resulting multiple R on a computing machine. *J. gen. Psych.*, 1933, **8**, 278-281.
7. Chauncey, Henry. A method for solving simultaneous equations, with particular reference to multiple correlation problems. *Harvard educ. Rev.* 1939, **9**, 63-68.
8. Deming, H. G. A systematic method for the solution of simultaneous linear equations. *Amer. math. Monthly*, 1928, **35**, 360-363.
9. Aitken, A. C. Studies in practical mathematics. I. The evaluation, with applications, of a certain triple product matrix. *Proc. Royal Soc., Edinburgh*, 1937, **57**, 172-181.
10. Doolittle, M. H. Method employed in the solution of normal equations and the adjustment of a triangulation. *U. S. Coast and Geodetic Survey Report*, 1878, 115-120.
11. Waugh, F. V. A simplified method of determining multiple regression constants. *J. Amer. Stat. Assoc.*, 1935, **30**, 694-700.
12. Kurtz, A. K. The use of the Doolittle method in obtaining related multiple correlation coefficients. *Psychometrika*, 1936, **1**, 45-51.
13. Tolley, H. R. and Ezekial, Mordecai. The Doolittle method for solving multiple correlation equations versus the Kelley-Salisbury iteration method. *J. Amer. Stat. Assoc.*, 1927, **22**, 497-500. (Particularly valuable for bibliography.)
14. Kelley, T. L. and McNemar, O. Doolittle versus Kelley-Salisbury methods

for computing multiple regression coefficients. *J. Amer. Stat. Assoc.*, 1929, 24, 164-169.

15. Rider, P. R. An introduction to modern statistical methods. New York: John Wiley and Sons, 1939.
16. Fisher, R. A. Statistical methods for research workers. Fifth Edition. London: Oliver and Boyd, 1934.
17. Dwyer, P. S. The simultaneous computation of groups of regression equations and associated multiple correlation coefficients. *Annals math. Stat.*, 1937, 8, 224-231.
18. Horst, Paul. A short method for solving for a coefficient of multiple correlation. *Annals math. Stat.*, 1932, 3, 40-44.
19. Horst, Paul. A general method for evaluating multiple regression constants. *J. Amer. Stat. Assoc.*, 1932, 27, 270-278.
20. Dunlap, J. W. Workbook in statistical method. New York: Prentice Hall, Inc., 1939. (Note bibliography.)
21. Tucker, L. R. A method for finding the inverse of a matrix. *Psychometrika*, 1938, 3, 189-197.
22. Gauss, C. F. Supplementum theoriae combinationis observationum erroribus minimis obnoxiae. Werke, Band 15, 1873.
23. Schott, C. A. The solution of normal equations by indirect elimination. *U. S. Coast and Geodetic Survey Report*, 1855, 255-264.
24. Carr, G. S. A synopsis of elementary results in pure mathematics. London: Francis Hodgson, 1886, p. 42.
25. Merriman, M. Text book in method of least squares. New York: John Wiley and Sons, 1884, 51-59.
26. Whittaker, E. T. and Robinson, G. The calculus of observations. London: Blackie and Son, 1924, 231-239.
27. Wren, F. L. Neo-Sylvester contractions and the solution of systems of linear equations. *Bull. Amer. math Soc.*, 1937, 43, 823-834.
28. Wherry, R. J. A modification of the Doolittle method: A logarithmic solution. *J. educ. Psych.*, 1932, 23, 455-459.

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ITEM SYNONYMIZATION: A METHOD FOR DETERMINING THE TOTAL MEANING OF PENCIL-PAPER REACTIONS

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Items have been studied heretofore for their value as elements of particular tests to the neglect of more fundamental research into the multiple potentiality of items. This article proposes a method of grouping items into "synonymies" comprising all of the items which correlate with a given key item. These synonymies can be used for interpretation of the total meaning of the key item: (1) by inspection of the constituent items and (2) by correlational study of obtained single scores of individual persons. The method is illustrated by four items with inter- and intra-correlations, and characteristics of an ideal background reservoir of items are pointed out.

Since items are the stuff of which tests are made, item analysis, interpretation, and understanding should rank high in the general technology of test construction. Yet this field of study has received only limited attention.

Research published within the past four years includes at least fifteen studies* dealing with the selection, validation, and evaluation of test items. An inspection of these studies reveals a serious effort to advance the technology of test construction, but the emphasis is placed upon the appraisal of items as elements of tests to the neglect of more fundamental analysis to determine the elements of items. Thus, current methods consist mostly of means of correlating items with total score or with a single external criterion. This is done for the purpose of determining the item's contribution to the measurement of a particular variable rather than for determining the item's multiple potentiality. Investigators, apparently being in haste to find whether an item is meaningful and suitable for the purpose of a particular test being constructed, after finding the item correlating or

* Articles appear under these names: Dorothy C. Adkins and H. A. Toops, Walter Merrill, J. C. Flanagan, T. J. Kelley, G. F. Kuder, T. F. Lentz, J. Lev, L. H. Mathews, C. J. Mosier and J. V. McQuitty, R. C. Pace, M. W. Richardson and Dorothy C. Adkins, E. A. Rundquist, R. F. Sletto, M. Smith, F. Swineford.

not correlating, have no further interest in the item. They are not interested in determining all the traits for the measurement of which the item is valuable, i.e., all the variables with which it is correlated.

Assumptions regarding the value of an item for a particular purpose are proved fallacious whenever the item is not found to correlate with the variable for which the test is being constructed. In all such instances it would seem natural to raise questions as to what variable or variables the item would predict. This regard for multiple potentiality has been evidenced to a very slight degree by Sletto* and by Lentz.† Sletto found that several pencil-and-paper items originally placed in a morale scale discriminated better in another scale. The morale item, "Most people can be trusted," yielded its highest discriminative value in a law scale. For the purposes of his study, this discovery was sufficient, and he did not attempt to account for the rest of the variance in the item. Lentz observed the variance among pencil-and-paper reactions in two fields: government and religion. He noted that the religious item, "In Sunday School chiefly the Bible should be taught," showed a higher correlation with a government battery than did the item, "The ballot should not be extended below the age of twenty-one," which appeared logically as a governmental item. As was the case with Sletto, Lentz did not go beyond the concern with correlating with one or two predetermined variables.

The necessity for this wider approach to the analysis of items is appreciated by Sheviakov and Friedberg,‡ who state:

> Any widespread use of these questionnaires must be postponed until more exact meanings of the items to the people responding have been accumulated.

Other persons within and without the field of measurement have expressed dissatisfaction regarding current pencil-and-paper items. Most of these criticisms, however, are so much "without form" as to be almost "void" from a constructive point of view. The writers of this article believe that these dissatisfactions relate to vital though poorly understood deficiencies in our test technology. The free expression of these dissatisfactions represents reason for hope for improvement. True and intensive item analysis should lead, through better appreciation of item deficiency, to the creation of better items.

* R. F. Sletto. Construction of personality scales by the criterion of internal consistency. Minneapolis, Minnesota: The Sociological Press, 1937.

† T. F. Lentz. Generality and specificity of conservatism-radicalim, *J. educ. Psychol*, 1938, 10, 540-546.

‡ G. V. Sheviakov and Jean Friedberg. Use of interest inventories for personality study. *J. educ. Research*, 1940, 33, 692-697.

PURPOSE

The purpose of this article is to report experiments with a method known as "item synonymization." Item synonymization aims to determine, borrowing a phrase from the foregoing quotation, "more exact meanings of the items to the people responding." It seeks as a goal the *total* meaning of pencil-and-paper items. In contrast to other methods, it is not concerned with the question, is the particular item related to a specific variable? Rather it is concerned with the question, what are the variables with which the item is related? Current methods attempt to find how much the item correlates with a proposed criterion; item synonymization seeks to find all of that with which the item correlates. Current methods attempt to measure an item in terms of one aspect or one dimension; synonymization attempts to identify and measure its various dimensions.

Reaction to an item is believed to be the net resultant of several variables, unitary or otherwise, such as conservatism, social-mindedness, self-confidence, acquiescence, or tolerance. These components are not easily detected in the specific reactions to a single item in a highly restrictive setting, but it seems reasonable to expect them to appear in a group of related items when examined in the aggregate. Consequently, item synonymization is promulgated on the assumptions that the symptomatic value of a pencil-and-paper item can be determined by considering the item in relation to other items, and that correlated items as a group give a more adequate picture of its symptom-bearing components than the item itself. The group of related items, by virtue of its nature and use, is designated the *item synonymy*.* An item synonymy can be more explicitly defined as a group of items consisting of all the items, or an adequate sample thereof, which are found to correlate with a single key item and used to secure single scores upon individual persons.

METHOD

Item synonymization consists of the following steps:

1. Securing reactions of many subjects to many items;
2. Correlating a particular item — the key item — with each of the other items;
3. Selecting appreciably correlated items for an item synonymy;
4. Inspecting these correlated items for purposes of interpreting the meaning of the key item;

* Commonly called "battery,'" but synonymy is more descriptive. As one refers to the synonymy or list of synonyms of a word to find all of its meanings, may he not also refer to an item synonymy to find the item's meanings?

5. Scoring each member of the old, or a new, population, using the synonymy as a test, and testing for reliability;
6. Correlating this score with other scores of known meaning;
7. Correlating the item and its synonymy;
8. Comparing the item correlates with the synonymy correlates.

Experiments with this method of item analysis are being made with data from the *Youth Expressionaire,* compiled by the Character Research Institute, Washington University, in cooperation with the United States Office of Education, Washington, D.C.* This instrument contains three thousand pencil-and-paper items, including sixteen testing techniques: opinionaire; word cross-out; picture rating; preferences in occupations, personages, food, books, quotations, jokes, and traits of friends; activities liked and engaged in; word-association; personality self-rating; slogan recognition; matching famous names; and status facts. The instrument is self-administering and requires, on the average, seven hours to complete. The reactions of approximately one thousand persons of various walks of life, from widely scattered parts of the United States, have been obtained. Ninety-five per cent of the cases range in age from seventeen to thirty, and all have had at least a high school education.

Method Illustrated

For illustration of the first four steps of the procedure, synonymization of the key item, "Do you read poetry frequently, occasionally, or never?," is herewith presented. As time did not permit the use of all the data available from the *Youth Expressionaire,* an instrument area of 479 opinion items and a population group of three hundred white persons were selected for study. For purpose of contrast, the population was separated into two groups, one group comprising those who said they frequently read poetry and the other group comprising those who said they never read poetry. All forthcoming deductions in regard to the behavior of items will be in terms of differences between these two groups.

In correlating the key item with each of the 479 other items, percentage differences were used as the measure of closeness of relationship. For example, it was found that 24 per cent of the group who said they frequently read poetry and 50 per cent of the group who said they never read poetry, reacted positively to the item, "I dislike radicals of any kind." (See Table 1) Subtracting 50 per cent

* For some of the data the authors are indebted to the Work Projects Administration, Recreation Division, District No. 10, St. Louis, Missouri.

from 24 per cent, we secure a difference value of −26. Since this difference is four times the standard error of the difference, we accept this item (regarding radicals) as a negative correlate of the item being analyzed. Out of 479 items studied, seventy-one showed a difference value of at least twice the standard error of the difference. From these there were chosen sixty (thirty positively and thirty negatively correlated) items to make up the poetry item synonymy. A few representative items from this synonymy are given in Table 1.

TABLE 1
Selections from Synonymy *P*

Expressionaire differences between those who frequently and those who never read poetry.

Column FRP Percentage of those who frequently read poetry reacting positively to each listed item.
Column NRP Percentage of those who never read poetry reacting positively to each listed item.
Column PD Percentage difference between the two groups.

FRP	NRP	PD	Item
45	26	+19	I would be willing to go to a social dance attended by both white and negro students.
61	43	+18	Do you laugh at yourself when things go wrong?
86	69	+17	I think it would be interesting to have some Chinese friends.
71	57	+14	Should all nations be members of a World Court or League of Nations to arbitrate their differences?
24	50	−26	I dislike radicals of any kind.
15	36	−21	Are you often made unhappy by not being able to keep up with the group?
33	53	−20	Do you think that modern society are only a bunch of "stuffed shirts"?
30	49	−19	Segregation of nationalities of different colors should be encouraged.

Upon preliminary inspection certain meanings appear in the items in the synonymy. Thus, the first item in the foregoing table apparently indicates tolerance; the second, happiness; the third and fourth, international-mindedness. In the minus list we note intolerance, unhappiness, cynicism, and non-international-mindedness. An inspection of the entire synonymy disclosed that the traits apparent in this table consistently recurred in other items in the same list with these traits. Since the items in the plus list were agreed to by a larger percentage of those who said they frequently read poetry than of those who said they never read poetry, it can be said that the meanings apparent in the items in the plus list are to some extent synonymous in meaning with a "frequently" reaction to the key item.

Similarly, it can be said that the meanings apparent in the minus list are to some extent synonymous in meaning with a "never" reaction to the key item. Hence, the key item can be synonymized in this way — those who say that they frequently read poetry are inclined to be tolerant, international-minded, and happy; those who say that they never read poetry tend to be intolerant, non-international-minded, cynical, and unhappy.

It seems to the writers that the consistency with which one simple pencil-and-paper item can separate people into two groups, as was illustrated, is significant, if not amazing. The question, "Do you read poetry frequently, occasionally, or never?," may have seemed to some of the reactors a senseless question to ask, but surely experiments with items like this should lead to a better understanding of character when it can be shown that the items on one list (of the positively correlated items) in the synonymy clearly reveal a trait which does not appear directly, and therefore not contradictorily, but conversely, and hence corroboratively, in the other list (of the negatively correlated items). This consistency gives an insight into the vast possibilities of item synonymization for the test maker. If an item clearly differentiates the reactors, it can be considered a strong item, however senseless on the surface it may seem. May we not say that an item clearly selects and differentiates reactors when two such selected and differentiated groups of reactors can be used in turn to select and differentiate groups of items so consistently?

CORRELATIONS*

An item synonymy has been shown to be useful for purposes of a type of inspection which discloses meanings of the item not apparent without the synonymy. A synonymy, however, may be further studied by treating it as a test unit, yielding for each subject a score suitable for intra-and inter-score correlation.

Certain questions are in order. Will the synonymy yield a reliable score? To what degree can the synonymy thus scored measure the same continuum as the item itself? In answer to the latter question, which concerns the correlation between the item and the synonymy, very limited data are here presented for synonymy scores obtained on the following items:

(l) Telling a lie is worse than taking the name of God in vain.
(s) Do you smoke?
(d) Would you like to be a playground director?
(p) Do you frequently, occasionally, or never read poetry?

* A full account is given by Edith Whitmer, "Item Analysis Through Correlations," Unpublished Master's Thesis, Washington University, 1940.

Hereafter these items will be referred to as *l*, *s*, *d*, and *p*, and their corresponding synonymies as *L*, *S*, *D*, and *P*, respectively.

Based upon the reactions* of one hundred subjects, the following median correlations (uncorrected) were found:

1. Synonymy reliabilities (split-half corrected) - .72 (4)†
2. Items with own synonymies - - - - - - - - - .53 (4)
3. Items with own synonymies (corrected for attenuation)†† - - - - - - - - - - - - - .82 (4)
4. Items with items - - - - - - - - - - - - - - .11 (6)
5. Synonymies with synonymies - - - - - - - - .28 (6)
6. Items with synonymies, excluding their own - - .26 (12)

TABLE II
Item-Synonymy Intercorrelations
(uncorrected)

	L	P	S	D
l	.52	.24	.29	.36
p	.34	.56	.31	.35
s	—.06	.17	.53	.16
d	.14	.20	.28	.51

TABLE III
Item Synonymy Intercorrelations
(corrected for attenuation)

	L	P	S	D
l	.71	.38	.49	.53
p	.47	.89	.53	.51
s	—.08	.27	.89	.24
d	.19	.32	.48	.77
Rel. Coef.	.88	.66	.58	.77

Accepting the oft-quoted definition that reliability is the degree to which the test measures whatever it measures, and assuming that a test is any group of items that measures (reliably) anything, we can say that we are statistically justified in considering the synonymy as a test, since it yields a single score of ascertainable and appreciable reliability.

* Reactions to 959 other items were used in the construction of *D* and 479 for each of the other synonymies.

† Numbers in parentheses refer to the number of coefficients represented by the median.

†† For attenuation due to unreliability of both synonymy and item. The reliability of the item was assumed to be .60. (See Lentz, *op. cit.*; also Lentz, T. F., Reliability of opinionaire technique studied intensively by the retest method, *J. soc. Psychol.* 1934, 5, 338-364.

Our second question pertains to the validity of the score. What is the synonymy measuring? This is not so easily answered. Most important here is the correlation between the item and the synonymy which purports to represent it. These correlations, as shown in Table 3, ranging from .71 to .89 when corrected for attenuation, indicate that we are on our way towards the goal of developing a synonymy that represents the same variance as that represented by the item. The fact that we have not fully arrived, however, is very clear, inasmuch as our item synonymy correlations are falling considerably short of 1.0.

A further reflection upon the trend of the development of the synonymy is found in the contrast between .11, our median inter-item correlation, and .28, our median inter-synonymy correlation. If we were merely measuring the same thing with more reliability, it is not likely that the inter-synonymy correlation would rise as high as .28. We would have to assume that the true reliabilities of the items were around .30 for the shift from .11 to .28 to be accounted for merely on the basis of change in reliability. Since, however, the reliabilities of the items are much more likely to be around .60, a better hypothesis would be that part of this increase in intercorrelation between synonymies to .28 is due to a common factor which has spuriously entered the various synonymies. The median intercorrelation of .26 between the item and the synonymy of another item points in the same direction. This common spurious factor can be due to defects in the common background reservoir of 479 items from which we have found our correlated items to make up our synonymies. Further experiments with certain other item synonymies, too lengthy for detailed presentation here, give weight to this last hypothesis.

These considerations sharpen our curiosity to know whether a synonymy can be developed which will correlate perfectly (unreliability allowed for) with the key item. What are the characteristics of a background reservoir of items which make this most feasible? How large a number of items — what types and varieties? Furthermore, aside from the limitations of the background reservoir of items, in this case the Youth Expressionaire which was only partially used, is it theoretically possible to develop a synonymy representing totally the variance of a single item? Is there something specific in each item which could never be measured by any synonymy? For example, a correlation of .89, as represented in Table 3, might be thought of as representing the communality between Item p and its synonymy of items, and the difference between this and unity as representing the specificity of Item p. Does this unmeasurable specificity, if it exists, lie within or without the concept of the unreliability of the item?

Further experimentation with the type of data now available should very rapidly clear up these questions.

SUMMARY

Item synonymies (groups of correlated items) are proposed as a much needed step in the field of item analysis. The term "item synonymization" is proposed as a technical label for the process of developing for a given item a synonymy, useful by way of inspection and correlation, for throwing more light on the nature of an item.

The experimentation here reported on four items yields interesting inspective interpretations as well as significant intra- and intercorrelations. An r of .89, estimated on the basis of certain assumptions, between the item and its synonymy is secured under conditions which make it probable that (a) a degree of impurity resides in the synonymy and (b) that a more adequately representative synonymy can be obtained by the use of a greater reservoir from which to pick correlated items.

Generally, our conclusion is that item synonymization possesses rich possibilities for item analysis and the extension of test item technology and that this or some other correlational method of intensive study will yield rich returns in more adequate pictures of what items represent and measure.